

Classification of some graded not necessarily associative division algebras I

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Abstract

We study not necessarily associative (NNA) division algebras over the reals. We classify those that admit a grading over a finite group G , and have a basis $\{v_g | g \in G\}$ as a real vector space, and the product of these basis elements respects the grading and includes a scalar structure constant with values in $\{1, -1\}$. That is, the algebra has a twisted product, and it is a twisted group algebra. We classify those graded by a group G of order $|G| \leq 4$. We will find the complex, and quaternion algebras, but also a remarkable set of novel non-associative division algebras.

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1 Introduction

We initiate the classification of not necessarily commutative and not necessarily associative division algebras over \mathbb{R} in which every nonzero element has a left and a right inverse (and thus no zero divisors), but the left and right inverses do not necessarily coincide. Accordingly, the inverses are not necessarily two-sided. We will recover in the process the complex, quaternion, and octonion division algebras and a new remarkable set of non-commutative and non-associative algebras over the reals.

Definition 1. *A unital not-necessarily-associative ring R with multiplicative identity element 1 is called a **not-necessarily-associative (NNA) division ring** if for every $v \neq 0$, $v \in R$ the left-multiplication map $x \mapsto v \cdot x$, and the right-multiplication map $x \mapsto x \cdot v$ are bijections. From this it follows that for each non-zero element v in R there exist elements $LI(v)$ and $RI(v)$ in R such that $LI(v) \cdot v = v \cdot RI(v) = 1 \in R$. The elements $LI(v)$ and $RI(v)$ are called the **left- and the right- inverse of v respectively**. If additionally for each $v \neq 0$ in R , $LI(v) = RI(v)$, then R is called a not-necessarily-associative division ring **with two-sided inverses**. Otherwise it is called a not-necessarily-associative division ring **with chiral inverses**. If the not-necessarily-associative ring with its summation and product is also an algebra over a field K we arrive at the concept of **not-necessarily-associative (NNA) division algebra over a field K** .*

The not-necessarily-associative (NNA) division algebras will be instrumental in the study of the representation theory of some non-associative structures, and we give this as an initial motivation for the classification we are going to pursue:

Definition 2. *Let $A \subset K^* = K - \{0\}$ be a multiplicative subgroup of K^* , with K a field, and let G be a finite group of order $|G|$ with identity element e . Given a function*

$$C : G \times G \rightarrow A \subset K^*,$$

*we call C a **structure constant of G in A** . We say that C is **unital** if additionally*

$$C(e, g) = C(g, e) = 1 \quad \forall g \in G.$$

We will frequently present the structure constant as an array of numbers in A , with matrix labels in G .

Definition 3. *Let C be a unital structure constant of G in A . We define a **twisted group**, denoted $A \times_C G$, as the set $A \times G$ endowed with the not-necessarily-associative binary operation (product):*

$$(\alpha, g) \cdot (\beta, h) = (\alpha\beta C(g, h), gh). \quad (1)$$

Every twisted group is a loop (quasi-group with identity $(1, e)$). We will study here the case where $K = \mathbb{R}$ and $A = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z} \equiv \mathbb{Z}_2$. In this

case, the twisted group $A \times_C G$ has $2|G|$ elements, and it constitutes a non-necessarily-associative generalization of discrete finite groups (which builds a particular and most prominent sub class of the twisted groups). The representation theory of twisted groups and some not-necessarily-associative semisimple algebras leads to the study and classification of a certain kind of not-necessarily-associative division algebras, which will be the scope of this paper series.

There are multiple equivalent definitions of twisted group algebras over a field K when the grading group is finite. We are interested on classifying some of them with particular properties, so we provide a definition that emphasizes such properties.

Definition 4. *Let G be a finite group with identity element e . A **twisted group algebra \mathcal{A} over K** is a K -algebra with unit $1 \in \mathcal{A}$ that as a vector space over K can be decomposed as*

$$\mathcal{A} = \bigoplus_{g \in G} W_g$$

$$\text{where } \dim_K W_g = 1, \text{ and } W_g \cdot W_h \subset W_{gh}.$$

and for every choice of base $v_g \in W_g$ for each $g \in G$, $g \neq e$, and $v_e = 1$, there exists a unital structure constant C for G in K^* , so that $v_g \cdot v_h = C(g, h)v_{gh}$, for all $g, h \in G$.

We will present now a definition that emphasizes the existence of structure constants with remarkable characteristics: let C be a unital structure constant of G in $A \subset K^*$. Let $\mathcal{A} = K_C G$ be a unital graded not-necessarily-associative algebra over a field K , which as a vector space over K has a basis $\{v_g : g \in G\}$, and has a C -twisted product, that is a product which extends bi-linearly from the product of basis elements

$$v_a \cdot v_b = C(a, b) v_{ab}, \quad \forall a, b \in G. \quad (2)$$

We call such an algebra $\mathcal{A} = K_C G$ or $(\mathcal{A}; G, K, A, C)$ a **twisted group algebra over K** . Clearly, the multiplicative neutral element of \mathcal{A} is v_e . The group G will be called the **grading group of $\mathcal{A} = K_C G$** . In the case where C is constant, that is $C(a, b) = 1$, $\forall a, b \in G$, the algebra \mathcal{A} coincides with the group algebra KG .

Observe that our definition highlights the existence of a basis for which the structure constant has particular properties: it takes values in $A \subset K^*$. We want to classify the twisted group algebras which are not-necessarily-associative division algebras. Concretely, we want to classify the twisted group algebras $(\mathcal{A}; G, K, A, C) = (\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ which are not-necessarily-associative division algebras over the reals. Observe that we have a finite grading group G and the existence of a basis in which the structure constant can only take values in $\{1, -1\}$. This makes this classification a finite problem, since there are only finitely many structure constant choices to make.

We remark also that although the twisted group algebra has a natural grading structure, we will classify the algebras using **plain \mathbb{R} -algebra isomorphisms** as far as we can. There are of course graded algebra isomorphisms

which are more restrictive since they need to respect fixed grading assignments. We will address those in [20].

H. Hopf proved in [1] that every not-necessarily-associative finite-dimensional division algebra over \mathbb{R} has dimension a power of 2. We will reproduce in an elementary fashion this result for the twisted group algebras we are considering. M. Kervaire in [2] and R. Bott and J. Milnor in [3] independently proved that every not-necessarily-associative finite-dimensional division algebra over the reals has dimensions 1, 2, 4, or 8. For some recent developments on NNA division algebras and twisted group algebras see [4], [5], [6], [7], [8], [6], [9], [10], [11], [12], [13], [14], [15], [16], [17], and [18]. In particular, we complement with this paper series early classifications using some generalizations of the Cayley–Dickson doublings in [8], [7], [4] in the case where the NNA division \mathbb{R} -algebra is a twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ with no constraint on the type of generalized Cayley–Dickson doubling. We will single out some NNA division \mathbb{R} -algebras among the ones already found by other authors, and we study some of their astonishing properties. But we will also find some novel NNA division \mathbb{R} -algebras which are not among the families previously studied.

We begin by exploring the zero divisors in a generic twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$, we give then some cohomological characterization for their non-associativity or non-commutativity, and define standard bases for those algebras as vector spaces over \mathbb{R} . Then we proceed by the order of the grading group classifying the corresponding not-necessarily-associative division algebras over \mathbb{R} . This paper will address the classification of twisted group algebras $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ which are not-necessarily-associative division algebras over \mathbb{R} with grading group of order lower or equal to 4. We will also examine the Lie- and Jordan-admissibility of the novel algebras, and will give a preliminary exploration of their deformations. In a following contribution [19] we classify those graded by the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$. In a further contribution [20] we classify those graded by the dihedral group D_4 as well as those graded by the quaternion group. A final contribution [21] will classify those graded by \mathbb{Z}_8 .

2 Zero divisors in twisted group algebras

Let $x, y \in (\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$, such that

$$x = \sum_{a \in H \subset G} x_a v_a, \quad y = \sum_{b \in H \subset G} y_b v_b, \quad (3)$$

where H is a subgroup of G . We call the real coefficients x_a and y_a the components of x and y respectively. Observe that the components of x and y are zero for basis elements v_a for $a \notin H$.

The product $x \cdot y$ is given by:

$$\begin{aligned} x \cdot y &= \left(\sum_{a \in H} x_a v_a \right) \cdot \left(\sum_{b \in H} y_b v_b \right) = \sum_{a, b \in H} x_a C(a, b) y_b v_{ab} \\ &= \sum_{c, a \in H} [x_a (C(a, a^{-1}c) y_{a^{-1}c})] v_c \end{aligned} \quad (4)$$

$$= \sum_{c, b \in H} [(x_{cb^{-1}} C(cb^{-1}, b)) y_b] v_c. \quad (5)$$

From the product of basis elements (2) it is clear that the product of x by y can have non-zero components only for basis elements with labels in H . That is, the set of elements in \mathcal{A} of the form (3) for H subgroup of G constitute automatically a subalgebra which is H -graded. Clearly, a zero divisor in a subalgebra implies a zero divisor in the whole algebra. In the case $H = G$, the subalgebra is the whole algebra \mathcal{A} .

From the product in (4-5) there are zero divisors if we obtain non-trivial (component) solutions to the systems

$$\begin{aligned} \sum_{a \in H} [C(a, a^{-1}c) y_{a^{-1}c}] x_a &= \sum_{a \in H} (M^L)_{c,a} x_a = 0, \\ \sum_{b \in H} [x_{cb^{-1}} C(cb^{-1}, b)] y_b &= \sum_{b \in H} (M^R)_{c,b} y_b = 0, \end{aligned}$$

where

$$\begin{aligned} (M^L)_{c,a} &= C(a, a^{-1}c) y_{a^{-1}c}, \\ (M^R)_{c,b} &= x_{cb^{-1}} C(cb^{-1}, b). \end{aligned}$$

Accordingly, there are zero divisors if there are non-trivial solutions to either of the equations:

$$\det(M^L)_{c,a} = \det[C(a, a^{-1}c) y_{a^{-1}c}]_{c,a} = 0, \quad (6)$$

$$\det(M^R)_{c,b} = \det[x_{cb^{-1}} C(cb^{-1}, b)]_{c,b} = 0. \quad (7)$$

We review now two basic facts:

- (i) A group G with order $|G|$ divisible by a prime p has always a subgroup of order p , and thus it has elements of order p . This is Cauchy's Theorem, which is in turn a special case of Sylow's Theorem.
- (ii) A polynomial $P(s)$ of odd degree in a single variable s has always a real root.

Let us assume that $|G|$ divisible by a prime p . From the first fact we conclude that a twisted group algebra $\mathcal{A} = \mathbb{R}_C G$ has a twisted group subalgebra with grading group that we call H , which is generated by an element $g \in G$ of order p . \mathcal{A} has thus a subalgebra with grading group H with $|H| = p$.

The determinants in (6-7) are homogeneous of order $|H|$ in the non-zero components involved, and is of order $|H|$ in each of the nonzero components. We adopt $y_g = x_g = 1$ for all $g \neq e$, $g \in H$, and then the left-hand side of (6-7) becomes polynomials of order $|H|$ in a single variable y_e and x_e respectively. Furthermore, the coefficients for the monomials of highest order $y_e^{|H|}$ and $x_e^{|H|}$ have absolute value $|\prod_{a \in H} C(a, e)| = |\prod_{b \in H} C(e, b)| = 1$, since the structure constant is unital. Hence, with the given choices, a twisted group subalgebra with grading group generated by a $g \in G$ of **odd order** p will lead to determinants which are polynomials of odd order in y_e , respectively in x_e , and thus it has nontrivial zero divisors according to the second basic fact (for the non trivial choice $y_g = x_g = 1$ for all $g \neq e$, $g \in H$, and y_e and x_e a real root of the odd degree polynomial, and $y_g = x_g = 0$ for all $g \notin H$). Since the reals viewed as an \mathbb{R} -algebra is graded by the trivial group $G = \{e\}$, with $|G| = 2^0 = 1$, we just proved the following proposition. This is a particular case of Hopf's theorem but in this instance it is obtained with very elementary tools:

Proposition 1. *A twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ with $|G|$ not a power of 2 has zero divisors.*

Clearly, this proposition can be extended for $A \subset K^* = K - \{0\}$, with A an arbitrary subgroup of K^* , the multiplicative group of a field K .

A question arises now: Are all the not-necessarily-associative division algebras over \mathbb{R} twisted group algebras over \mathbb{R} ? Equivalently, we ask if there is a not-necessarily-associative division ring which is not compatible with some G -grading, a basis $\{v_g | g \in G\}$ as vector space, and a graded product (2) at the level of basis elements? We will address this question in [21] and [41].

We finish this section establishing a necessary and sufficient condition for non-necessarily-associative division algebra in the finite dimensional case.

Proposition 2. *Let \mathcal{A} be a finite dimensional algebra over \mathbb{R} . The algebra \mathcal{A} is a non-necessarily-associative division algebra if and only if it has no non-trivial zero divisors.*

Proof. Let \mathcal{A} be a finite dimensional not-necessarily-associative division algebra. So, \mathcal{A} is finite dimensional as a vector space over \mathbb{R} . Let c be a non-zero element $c \in \mathcal{A}$. The function product from the left $x \mapsto c \cdot x$ and the function product from the right $x \mapsto x \cdot c$ are both linear transformations (since the product is bilinear). Now, they are both bijections then their kernels are trivial ($\{0\}$). Accordingly, the products $c \cdot x$ and $x \cdot c$ are zero only when $x = 0$. So, there are no non-trivial zero divisors. Conversely, assume that \mathcal{A} is finite dimensional and has no non-trivial zero divisors. Let c be a non-zero element $c \in \mathcal{A}$. Since there are no non-trivial zero divisors, then the kernels of the linear transformations product from the left $x \mapsto c \cdot x$ and product from the right $x \mapsto x \cdot c$ are both $\{0\}$. Since the algebra is finite dimensional then such transformations are injective and onto. \square

3 Cohomological Considerations

We will define certain functions that will characterize the non-associativity or non-commutativity properties of a twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ in terms of some of their cohomological properties.

Definition 5. Let $(\mathcal{A}; G, K, A, C)$ be a twisted group algebra. We define the r -function as a co-boundary (see [22], [23]) of the structure constant C :

$$\begin{aligned} r : G \times G \times G &\rightarrow A, \\ r(a, b, c) = (\delta C)(a, b, c) &= C(b, c) (C(ab, c))^{-1} C(a, bc) (C(a, b))^{-1}, \end{aligned}$$

where we assume a trivial action of G on A .

We perform triple products of basis elements using the twisted product (2), and find:

$$\begin{aligned} v_a \cdot (v_b \cdot v_c) &= r(a, b, c) (v_a \cdot v_b) \cdot v_c, \\ \mathcal{A} \text{ associative iff } r(a, b, c) &= 1, \forall a, b, c \in G. \end{aligned}$$

The condition for associativity is the same as the one required for having associativity in the product defined in (1) for twisted groups. In this case, the twisted group is actually a group. Obviously, if the structure constant C is itself a two coboundary, then the function r is constant, and the algebra is associative (see [14]). In some sense, the twisted groups and the twisted group algebras are a natural and close relaxation of the concepts of groups and group algebras, since although the function r is not necessarily constant, it is still not only a 3-cocycle, it is a 3-co-boundary.

Definition 6. Let $(\mathcal{A}; G, K, A, C)$ be a twisted group algebra. If G is abelian, we define the q -function:

$$q : G \times G \rightarrow A, \quad q(a, b) = C(a, b) (C(b, a))^{-1}.$$

Clearly,

$$\begin{aligned} v_a \cdot v_b &= q(a, b) v_b \cdot v_a, \\ \mathcal{A} \text{ commutative iff } q(a, b) &= 1 \forall a, b \in G. \end{aligned}$$

Definition 7. We say that the q function is a **2-cocycle** (see [22], [23]) iff

$$(\delta q)(g, h, t) = q(h, t) (q(gh, t))^{-1} q(g, ht) (q(g, h))^{-1} = 1 \quad \forall g, h, t \in G.$$

It turns out that if q is a 2-cocycle, then (see [24], [25], [26], [27]):

$$\begin{aligned} 1 &= (q(a, c) q(b, c) q(ab, c))^{-1} (r(a, b, c) r(c, a, b) r(b, c, a)), \\ 1 &= r(a, b, c) r(c, b, a), \quad \forall a, b, c \in G. \end{aligned} \tag{8}$$

Definition 8. We call q a **2-co-boundary** (see [22], [23]) iff there exists a function $\kappa : G \rightarrow A$, such that:

$$q(a, b) = (\delta\kappa)(a, b) = \kappa(b) (\kappa(ab))^{-1} \kappa(a).$$

We say that the function q is **separable** (see [25], [26], [27]) iff

$$q(h, t) (q(gh, t))^{-1} q(g, t) = 1 \quad \forall g, h, t \in G.$$

It turns out that, every separable q function is 2-cocycle. Besides the relation (8), a separable q -function satisfies a Jacobi-like identity (see [24], [25], [26], [27]):

$$r(a, b, c) r(c, a, b) r(b, c, a) = 1, \quad \forall a, b, c \in G. \quad (9)$$

4 Left- and right-standard bases

The next is a rather technical definition, but the resulting vector space bases are simple to handle and advantageous for profiting the knowledge about subalgebras. They also make simple Cayley–Dickson-like considerations. The reader might skip this section and return to it upon need.

In order to make considerations on twisted group algebras $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ which are not-necessarily-associative division algebras over \mathbb{R} , we will adopt certain families of bases as vector spaces over \mathbb{R} . As we will find, such bases can provide necessary and sufficient conditions for establishing \mathbb{R} -algebra isomorphism or non-isomorphism relationships between not-necessarily-associative division algebras graded by the same group.

Definition 9. Let $\{g_1, g_2, \dots, g_m\}$ be a minimal set of generators of the grading group G with identity element e of a twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$, whose orders are non-increasing: $O(g_1) \geq O(g_2) \geq \dots \geq O(g_m)$.

For all groups of order 1, 2, 4, or 8, which are not the quaternion group we have $|G| = O(g_1)O(g_2) \dots O(g_m)$, and every element $g \in G$ can be written uniquely as

$$g = g_m^{s_m} \dots g_1^{s_1}, \text{ where } 0 \leq s_i < O(g_i), \text{ and } g_i^0 = e, \ i = 1, \dots, m.$$

Every element $g \in G$ can also be written uniquely as

$$g = g_1^{c_1} \dots g_m^{c_m}, \text{ where } 0 \leq c_i < O(g_i), \text{ and } g_i^0 = e, \ i = 1, \dots, m.$$

The quaternion group has 8 elements but a minimal set of generators $\{g_1, g_2\}$ include two elements of order 4. Every element g of the quaternion group can be written uniquely in the form:

$$g = g_2^{s_2} g_1^{s_1}, \text{ where } 0 \leq s_1 < 4, \ 0 \leq s_2 < 2, \text{ and } g_i^0 = e, \ i = 1, 2.$$

Every element $g \in G$ can also be written uniquely as

$$g = g_1^{c_1} g_2^{c_2}, \text{ where } 0 \leq c_1 < 4, \ 0 \leq c_2 < 2, \text{ and } g_i^0 = e, \ i = 1, 2.$$

Notice that these particular expressions of the group elements in terms of a minimal set of generators are obvious for the abelian groups, and are easily verified for the two non-abelian groups of order 8: the dihedral group of 8 elements and the quaternion group. Associated with these **unique expressions of the grading group elements in terms of elements of a minimal set of generators** we define two types of standard bases:

(i) **A vector space basis** $\{v_g : g \in G\}$ for \mathcal{A} is called a **left-standard basis**, if $v_e = 1$, the unit element of the algebra, and there are arbitrary nonzero elements $v_{g_1}, \dots, v_{g_m} \in \mathcal{A}$ of pure degree g_1, \dots, g_m respectively, and for $g = g_m^{s_m} \cdots g_1^{s_1}$, the unique expression of g discussed above, we define

$$v_g = v_{g_m}^{s_m} \cdot (\cdots \cdot (v_{g_3}^{s_3} \cdot (v_{g_2}^{s_2} \cdot v_{g_1}^{s_1})) \cdots),$$

$$\text{where } v_{g_i}^0 \equiv v_e, \text{ and } v_{g_i}^{n+1} = v_{g_i} \cdot v_{g_i}^n, \forall 0 \leq n < O(g_i) - 1.$$

That is, new factors in a power are fed from the left. The basis elements are ordered using the associated m -tuples (s_1, \dots, s_m) . We say that $(s_1, \dots, s_m) < (t_1, \dots, t_m)$ when the last nonzero entry of $(s_1 - t_1, \dots, s_m - t_m)$ is negative. The first basis element is thus v_e which is associated to the m -tuple $(0, \dots, 0)$. The ordered standard basis elements for a group requiring more than one generator will be

$$[v_e \equiv 1, v_{g_1}, \dots, v_{g_1}^{O(g_1)-1}, v_{g_2}, v_{g_2} \cdot v_{g_1}, \dots, v_{g_2} \cdot v_{g_1}^{O(g_1)-1}, \dots].$$

This standard basis is called **normalized**, if additionally

$$v_{g_i}^{O(g_i)/2} \cdot v_{g_i}^{O(g_i)/2} \in \{1, -1\}, \text{ for all } 1 \leq i \leq m \text{ with } O(g_i) > 1.$$

Observe that since we are in a real algebra and all group orders are powers of 2, only one of the options can be satisfied. Furthermore, the basis element $v_{g_i}^{O(g_i)/2}$ generates a \mathbb{Z}_2 -graded subalgebra, and as we will see in the next section, in order to avoid zero divisors (and allow for a NNA division algebra), the only available normalization choice is $v_{g_i}^{O(g_i)/2} \cdot v_{g_i}^{O(g_i)/2} = -1$.

(ii) **A vector space basis** $\{v_g : g \in G\}$ for \mathcal{A} is called a **right-standard basis**, if $v_e = 1$, the unit element of the algebra, and there are arbitrary nonzero elements v_{g_1}, \dots, v_{g_m} of pure degree g_1, \dots, g_m respectively, and for $g = g_1^{c_1} \cdots g_m^{c_m}$, the unique expression of g discussed above, we define

$$v_g = (\cdots ((v_{g_1}^{c_1} \cdot v_{g_2}^{c_2}) \cdot v_{g_3}^{c_3}) \cdots) \cdot v_{g_m}^{c_m},$$

$$\text{where } v_{g_i}^{0^+} = v_e, \text{ and } v_{g_i}^{n+1^+} = v_{g_i} \cdot v_{g_i}^{n^+}, \forall 0 \leq n < O(g_i) - 1.$$

Observe that powers are defined differently for the right-standard basis, since new factors in a power are fed from the right. The basis elements are ordered using the associated m -tuples (c_1, \dots, c_m) . We say that $(c_1, \dots, c_m) < (t_1, \dots, t_m)$ when the last nonzero entry of $(c_1 - t_1, \dots, c_m - t_m)$ is negative. The

first basis element is thus v_e which is associated to the m -tuple $(0, \dots, 0)$. The ordered standard basis elements for a group requiring more than one generator will be

$$[v_e = 1, v_{g_1}, \dots, v_{g_1}^{O(g_1)-1}, v_{g_2}, v_{g_1} \cdot v_{g_2}, \dots, v_{g_1}^{O(g_1)-1} \cdot v_{g_2}, \dots].$$

The right-standard basis is called **normalized**, if additionally

$$v_{g_i}^{O(g_i)/2} \cdot v_{g_i}^{O(g_i)/2} \in \{1, -1\}, \text{ for all } 1 \leq i \leq m \text{ with } O(g_i) > 1.$$

Again, since we are in a real algebra and all group orders are powers of 2, only one of the options can be satisfied. The basis element $v_{g_i}^{O(g_i)/2}$ generates a \mathbb{Z}_2 -graded subalgebra, and as we will see in the next section, in order to avoid zero divisors (and allow for a NNA division algebra), the only available normalization choice is $v_{g_i}^{O(g_i)/2} \cdot v_{g_i}^{O(g_i)/2} = -1$.

Example: The abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has a minimal set of generators of two elements, both of order two. A right-standard basis for them will have the form

$$[v_e, v_{g_1}, v_{g_2}, v_{g_1} \cdot v_{g_2}],$$

whereas a left-standard basis will have the form

$$[v_e, v_{g_1}, v_{g_2}, v_{g_2} \cdot v_{g_1}].$$

We remark now, that if we have a twisted group algebra $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$, then we can always adopt a normalized left- (or right-) standard basis whose corresponding structure constant of G is in $\{1, -1\}$ (since not every basis change maintain this feature). The reason for that is simple. Assume we have a basis $\{\hat{v}_g | g \in G\}$ for \mathcal{A} with corresponding structure constant \hat{C} of G in $\{1, -1\}$. Let $\{g_1, \dots, g_m\}$ be a minimal generating set of G as in the previous definition. We adopt $v_e := \hat{v}_e = 1$ and $v_{g_i} := \hat{v}_{g_i}$, $i = 1, \dots, m$. Using these we can construct a normalized left- (or right-) standard basis as explained above, whose corresponding structure constant turns out to be also in $\{1, -1\}$. The reason for that is that the monomials with factors of the form \hat{v}_e or \hat{v}_{g_i} used to define the v_g (no matter which configuration of parenthesis) will be reduced to a product of structure constant factors $\hat{C}(g_i, g_j)$ or 1's (since the structure constant is unital) times a resulting \hat{v}_g . Hence, either $v_g = \hat{v}_g$ or $v_g = -\hat{v}_g$ occur. And the products $v_g \cdot v_h = C(g, h)v_{gh}$, will have structure constant that can differ from $\hat{C}(g, h)$ only by a sign. The normalization will maintain the one underlying the older basis, since $v_{g_i} = \hat{v}_{g_i}$. Accordingly, **adopting a normalized left- (or right-) standard basis will constitute no constraint of generality**.

Another comment is in order, that will become more evident when we work with concrete algebras: Let $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ be a twisted group algebra over \mathbb{R} . If C is the structure constant for G in $\{1, -1\}$ for a given normalized left- (right-) standard basis choice, where the labels in G are taken in the order associated with the ordered normalized left- (right-) standard basis, then the opposite or mirror algebra will have a structure constant C_M for G in $\{1, -1\}$

which is the transposed of the array C when using the corresponding normalized right- (left-) standard basis but with the order of the normalized left- (right-) standard basis. Observe nevertheless, that for nonabelian grading groups the group labels in the basis and in C_M for the opposite algebra are taken g^{-1} instead of g , using the group anti-isomorphism $g \mapsto g^{-1}$. Notice furthermore, that not every twisted group algebra (even if the grading group is abelian) needs to be \mathbb{R} -algebra isomorphic to its opposite or mirror algebra.

5 Grading Group of order 2

There is only one group of order 2, the abelian group \mathbb{Z}_2 . We analyze this case in a thorough way in order to fix notation and to study analogous structures in the further cases. We consider the unital structure constant array, with $\alpha \in \{1, -1\}$:

C	0	1
0	1	1
1	1	α

Table I: Structure constant of $G = \mathbb{Z}_2$ in $\{1, -1\}$

where we adopted $\mathbb{Z}_2 = (\{0, 1\}; +)$ additive for the grading (and component) labels. In this case, equations (6-7) become:

$$\begin{aligned} \det(M^L)_{c,a} &= y_0^2 - \alpha y_1^2 = 0, \\ \det(M^R)_{c,b} &= x_0^2 - \alpha x_1^2 = 0. \end{aligned}$$

In order to have absence of zero divisors we are forced to adopt $\alpha = -1$. Accordingly there is exactly one choice of structure constant of \mathbb{Z}_2 in $\{1, -1\}$ leading to absence of zero divisors. We obtain in this way the complex numbers by identifying $v_0 \equiv 1$ and $v_1 \equiv i$, since $i^2 = v_1 \cdot v_1 = v_1^2 = C(1, 1)v_0 = -1$. Observe that the obtained basis $[v_0, v_1]$ (we use brackets to emphasize the order of the basis elements) is a normalized left- and right-standard basis, and its associated structure constant for a normalized left- and right-standard basis for a twisted group algebra $(\mathcal{A}; \mathbb{Z}_2, \mathbb{R}, \{1, -1\}, C)$ is unique.

The left-inverse $LI(y)$ of y has components $LI(y)_0$ and $LI(y)_1$ satisfying:

$$M^L \begin{bmatrix} LI(y)_0 \\ LI(y)_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving for the unknowns we obtain for $y \neq 0$:

$$LI(y) = \frac{y_0}{y_0^2 + y_1^2} v_0 - \frac{y_1}{y_0^2 + y_1^2} v_1 = \frac{y_0}{y_0^2 + y_1^2} - \frac{y_1}{y_0^2 + y_1^2} i.$$

We obtain similarly the right inverse of x , whose components satisfy:

$$M^R \begin{bmatrix} RI(x)_0 \\ RI(x)_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

And solving we obtain again for $x \neq 0$:

$$RI(x) = \frac{x_0}{x_0^2 + x_1^2} v_0 - \frac{x_1}{x_0^2 + x_1^2} v_1 = \frac{x_0}{x_0^2 + x_1^2} - \frac{x_1}{x_0^2 + x_1^2} i.$$

In this case $RI(x) = LI(x)$, and then it is a (plain) division algebra over the reals. Furthermore, from the structure constant we obtain

$$x \cdot y = (x_0 y_0 - x_1 y_1) v_0 + (x_0 y_1 + x_1 y_0) v_1. \quad (10)$$

From this we verify that the division algebra is commutative which follows also from the symmetry of the structure constant array. A simple computation shows that the operation is also associative. Accordingly, the obtained division algebra is a field: the field of complex numbers \mathbb{C} . We will denote each element of the algebra \mathbb{C} in terms of its components, that is, as pairs of real numbers:

$$x := x_0 v_0 + x_1 v_1 = [x_0, x_1]_{\mathbb{C}}. \quad (11)$$

And thus, the product of \mathbb{C} in components becomes:

$$x \cdot y = [x_0, x_1]_{\mathbb{C}} \cdot [y_0, y_1]_{\mathbb{C}} = [x_0 y_0 - y_1 x_1, y_1 x_0 + x_1 y_0]_{\mathbb{C}}. \quad (12)$$

This corresponds to the first version of the Cayley-Dickson process. We can define a **complex conjugate**

$$\bar{x} = [x_0, -x_1]_{\mathbb{C}}.$$

which turns out to be an involution, and an anti-homomorphism with respect to the product:

$$\bar{\bar{x}} = x = [x_0, x_1]_{\mathbb{C}}, \quad \overline{x \cdot y} = \bar{y} \cdot \bar{x}.$$

This makes \mathbb{C} an $*$ -algebra. We define also a scalar function $x \mapsto |x|_{\mathbb{C}} \geq 0$ satisfying

$$|x|_{\mathbb{C}}^2 = x \cdot \bar{x} = \bar{x} \cdot x = [x_0^2 + x_1^2, 0]_{\mathbb{C}} = x_0^2 + x_1^2,$$

where we identify the element $[r, 0]_{\mathbb{C}} \in \mathbb{C}$ with the element $r \in \mathbb{R}$. It turns out that it verifies also the **Schwarz inequality**, and it verifies it with strict equality:

$$|x \cdot y|_{\mathbb{C}}^2 = |x|_{\mathbb{C}}^2 |y|_{\mathbb{C}}^2, \forall x, y \in \mathbb{C}.$$

We verify that scalar function is actually a norm, by verifying that it satisfies **positive homogeneity, the triangle inequality, and positive definiteness**:

$$\begin{aligned} |\alpha x|_{\mathbb{C}} &= |\alpha| |x|_{\mathbb{C}}, \quad \forall \alpha \in \mathbb{R}, x \in \mathbb{C} \quad (\text{positive homogeneity}), \\ |x + y|_{\mathbb{C}} &\leq |x|_{\mathbb{C}} + |y|_{\mathbb{C}}, \quad \forall x, y \in \mathbb{C} \quad (\text{triangle inequality}), \\ |x|_{\mathbb{C}} = 0 &\implies x = 0, \quad \forall x \in \mathbb{C} \quad (\text{positive definiteness}). \end{aligned} \quad (13)$$

In fact, algebras with a norm that satisfies strict Schwarz equality are called **normed NNA division algebras**. Accordingly, \mathbb{C} is a normed division algebra. Furthermore, the inverses can be written in terms of the conjugates and the norm:

$$LI(x) = RI(x) = \frac{1}{|x|_{\mathbb{C}}^2} \bar{x}.$$

In terms of generators and relations the complex field \mathbb{C} is characterized by

$$\langle v_0 \equiv 1, v_1 \equiv i \mid i \cdot i = -1 \rangle.$$

In this case the q - and r -functions are just constants:

$$q_{\mathbb{C}}(g, h) = r_{\mathbb{C}}(g, h, t) = 1, \quad \forall g, h, t \in G.$$

6 Grading Group of order 4

There are two possible grading groups of order four: $\mathbb{Z}_2 \times \mathbb{Z}_2$, and \mathbb{Z}_4 . The former will lead to the quaternions and the latter to a novel not-necessarily-associative division algebra with chiral inverses.

6.1 Grading group: the Klein Group

We use additive group notation for the grading group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Without loss of generality, we adopt for the twisted group algebra $(\mathcal{A}; \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{R}, \{1, -1\}, C)$ a normalized right-standard basis $[v_{(0,0)} \equiv 1, v_{(1,0)}, v_{(0,1)}, v_{(1,1)} \equiv v_{(1,0)} \cdot v_{(0,1)}]$. We have chosen a right-standard basis to obtain the standard notation for the resulting algebra. From equation (2) we have $C((1,0), (0,1)) = 1$. Such a twisted group algebra has three different \mathbb{Z}_2 -graded twisted group subalgebras, whose structure constants have to be as those for the complex field in order to avoid zero divisors. This forces $C((1,0), (1,0)) = C((0,1), (0,1)) = C((1,1), (1,1)) = -1$. Using such a basis for a NNA division \mathbb{R} -algebra \mathcal{A} , the unital structure constant array with $\alpha, \beta, \delta, \epsilon, \phi \in \{1, -1\}$ has to have the from:

C	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	1	1	1	1
$(1,0)$	1	-1	1	α
$(0,1)$	1	β	-1	δ
$(1,1)$	1	ϵ	ϕ	-1

Table II: Structure constant of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ in $\{1, -1\}$

From this, and adopting for simplicity the components to be labeled by the sub-indices 0, 1, 2, 3 instead of $(0,0), (1,0), (0,1), (1,1)$, respectively, we obtain:

$$M^L = \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & y_0 & \delta y_3 & \phi y_2 \\ y_2 & \alpha y_3 & y_0 & \epsilon y_1 \\ y_3 & y_2 & \beta y_1 & y_0 \end{bmatrix}.$$

The determinant of M^L for some particular values of the components becomes:

$$\begin{aligned}\det M^L|_{y_2=y_3=0} &= (y_0^2 + y_1^2)(y_0^2 - \epsilon \beta y_1^2), \\ \det M^L|_{y_1=y_3=0} &= (y_0^2 + y_2^2)(y_0^2 - \phi y_2^2).\end{aligned}$$

In order to avoid zero divisors we need to require $\epsilon = -\beta$, and $\phi = -1$. In a similar manner for other choices of only two non-vanishing components we obtain the constraints $\alpha = -1$, $\delta = -\alpha = 1$, $\beta = -1$. From this we obtain that there is a unique choice of structure constant of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ in $\{1, -1\}$ for the chosen normalized right-standard basis that could avoid zero divisors, namely,

$C_{\mathbb{H}}$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	1	1	1	1
(1,0)	1	-1	1	-1
(0,1)	1	-1	-1	1
(1,1)	1	1	-1	-1

Table III: Structure constant array of \mathbb{H}

Similarly, there is a unique choice of structure constant when a normalized left-standard basis (instead of a normalized right-standard basis) is adopted. This is the opposite algebra to \mathbb{H} that turns out to be isomorphic to \mathbb{H} itself, and has structure constant equal to the transpose of the array $C_{\mathbb{H}}$ obtained above.

The determinants of M^L and M^R for the algebra \mathbb{H} with structure constant $C_{\mathbb{H}}$ become:

$$\begin{aligned}\det M_{\mathbb{H}}^L &= (y_0^2 + y_1^2 + y_2^2 + y_3^2)^2, \\ \det M_{\mathbb{H}}^R &= (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2.\end{aligned}$$

The absence of zero divisors follows from the positive definiteness of the determinants. The product turns out to be noncommutative (non-symmetric array) and in terms of its components takes the form:

$$\begin{aligned}x \cdot y &= [x_0, x_1, x_2, x_3]_{\mathbb{H}} \cdot [y_0, y_1, y_2, y_3]_{\mathbb{H}} \\ &= [x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3, x_1 y_0 + x_0 y_1 - x_3 y_2 + x_2 y_3, \\ &\quad x_2 y_0 + x_3 y_1 + x_0 y_2 - x_1 y_3, x_3 y_0 - x_2 y_1 + x_1 y_2 + x_0 y_3]_{\mathbb{H}}.\end{aligned}$$

The left- and right- inverses are also obtained using analogous expressions as those above for the \mathbb{Z}_2 -graded algebras. They turn out to be two-sided. We use a conjugation analogous to the one used in the complex case:

$$\bar{x} = [x_0, -x_1, -x_2, -x_3]_{\mathbb{H}}.$$

It turns out to be also an involution, and an anti-homomorphism with respect to the product:

$$\bar{\bar{x}} = x = [x_0, x_1, x_2, x_3]_{\mathbb{H}}, \quad \overline{x \cdot y} = \bar{y} \cdot \bar{x}.$$

This makes \mathbb{H} also an $*$ -algebra.

We define also a scalar function $x \mapsto |x|_{\mathbb{H}} \geq 0$ satisfying

$$|x|_{\mathbb{H}}^2 = x \cdot \bar{x} = \bar{x} \cdot x = [x_0^2 + x_1^2 + x_2^2 + x_3^2, 0, 0, 0]_{\mathbb{H}} = x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

where, again, we identify the element $[r, 0, 0, 0]_{\mathbb{H}} \in \mathbb{H}$ with the element $r \in \mathbb{R}$. It turns out that it verifies also the **Schwarz inequality**, and it verifies it with strict equality:

$$|x \cdot y|_{\mathbb{H}}^2 = |x|_{\mathbb{H}}^2 |y|_{\mathbb{H}}^2, \forall x, y \in \mathbb{H}. \quad (14)$$

We verify also that this scalar function is actually a norm, by verifying that it satisfies:

$$\begin{aligned} |\alpha x|_{\mathbb{H}} &= |\alpha| |x|_{\mathbb{H}}, \quad \forall \alpha \in \mathbb{R}, x \in \mathbb{H} \quad (\text{positive homogeneity}), \\ |x + y|_{\mathbb{H}} &\leq |x|_{\mathbb{H}} + |y|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H} \quad (\text{triangle inequality}), \\ |x|_{\mathbb{H}} = 0 &\implies x = 0, \quad \forall x \in \mathbb{H} \quad (\text{positive definiteness}). \end{aligned}$$

Hence, \mathbb{H} is a normed division algebra. Furthermore, the inverse for $x \neq 0$ can also be written in terms of the conjugates and the norm:

$$LI(x) = RI(x) = \frac{1}{|x|_{\mathbb{H}}^2} \bar{x}.$$

This division algebra is called the quaternion algebra \mathbb{H} over \mathbb{R} . In terms of generators and relations the algebra \mathbb{H} can be given by;

$$\begin{aligned} < v_{(0,0)} \equiv 1, \quad v_{(1,0)} \equiv i, \quad v_{(0,1)} \equiv j, \quad v_{(1,1)} \equiv k \mid i \cdot i = j \cdot j = k \cdot k = -1, \\ i \cdot j = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, \quad k \cdot i = -i \cdot k = j >. \end{aligned}$$

The structure constant function, the q -function (since G abelian), and the r -function are given by (see [24], [26], [27]):

$$C_{\mathbb{H}}((n, m), (n', m')) := \exp\{-\pi i [nn' + m(n' + m')]\}, \quad (15)$$

$$q_{\mathbb{H}}((n, m), (n', m')) := \exp\{\pi i [nm' - n'm]\}, \quad (16)$$

$$k_{\mathbb{H}}((n, m)) := \exp\{\pi i [-nm]\}, \quad q_{\mathbb{H}} = \delta k_{\mathbb{H}}, \quad (17)$$

$$r_{\mathbb{H}}((n, m), (n', m'), (n'', m'')) := 1, \quad (18)$$

$$\text{for all } (n, m), (n', m'), (n'', m'') \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Accordingly, the quaternion algebra \mathbb{H} is associative. It has a q -function which is a 2-coboundary, and it turns out to be also separable.

We can write the element of \mathbb{H} in terms of pairs of complex numbers:

$$x = [x_0, x_1, x_2, x_3]_{\mathbb{H}} = ([x_0, x_1]_{\mathbb{C}}, [x_2, x_3]_{\mathbb{C}})_{\mathbb{H}} =: (X_L, X_R)_{\mathbb{H}}. \quad (19)$$

The product of this division algebra becomes

$$\begin{aligned} x \cdot y &= (X_L, X_R)_{\mathbb{H}} \cdot (Y_L, Y_R)_{\mathbb{H}} \\ &= (X_L \cdot Y_L - \overline{Y_R} \cdot X_R, Y_R \cdot X_L + X_R \cdot \overline{Y_L})_{\mathbb{H}}. \end{aligned} \quad (20)$$

This gives an updated version of the Cayley-Dickson construction.

6.2 Grading group: cyclic of order 4

A twisted group algebra $(\mathcal{A}; \mathbb{Z}_4, \mathbb{R}, \{1, -1\}, C)$ as a vector space has a basis $[v_0 \equiv 1, v_1, v_2, v_3]$. As an algebra, \mathcal{A} can be generated by a $v_1 \equiv w$. We can adopt without loss of generality $v_2 \equiv w \cdot w \equiv w^2$, $v_3 \equiv w \cdot (w^2) \equiv w^3$, where we adopted the definition of cubic powers with this particular configuration of parenthesis (stacking new factors from the left). The set $[v_0 \equiv 1, v_2 \equiv w^2]$ generates a \mathbb{Z}_2 -graded subalgebra. In order to avoid zero divisors, we need to require $v_2 \cdot v_2 = w^2 \cdot w^2 = -v_0$. Accordingly, using the normalized left-standard basis $[1, w, w^2, w^3]$ for a NNA division \mathbb{R} -algebra \mathcal{A} , the unital structure constant array with $\alpha, \beta, \delta, \epsilon, \phi, \omega \in \{1, -1\}$ has the form:

C	0	1	2	3
0	1	1	1	1
1	1	1	1	α
2	1	β	-1	δ
3	1	ϵ	ϕ	ω

Table IV: Structure constant of $G = \mathbb{Z}_4$ in $\{1, -1\}$

The matrix M^L takes the form

$$M^L = \begin{bmatrix} y_0 & \alpha y_3 & -y_2 & \epsilon y_1 \\ y_1 & y_0 & \delta y_3 & \phi y_2 \\ y_2 & y_1 & y_0 & \omega y_3 \\ y_3 & y_2 & \beta y_1 & y_0 \end{bmatrix}. \quad (21)$$

The determinant of M^L when two components are zero takes one of the following forms

$$\begin{aligned} &(\beta y_1^2 - \delta y_3^2)(-\epsilon y_1^2 + \alpha \omega y_3^2), \quad (y_0^2 + y_2^2)(y_0^2 - \phi y_2^2), \\ &y_0^4 - \epsilon \beta y_1^4, \quad y_0^4 - \alpha \delta \omega y_3^4, \quad -\phi y_2^4 - \epsilon \beta y_1^4, \quad -\phi y_2^4 - \alpha \delta \omega y_3^4. \end{aligned}$$

If we evaluate these determinants when all remaining components have value 1 we obtain:

$$(\beta - \delta)(-\epsilon + \alpha \omega), \quad 2(1 - \phi), \quad 1 - \epsilon \beta, \quad 1 - \alpha \delta \omega, \quad -\phi - \epsilon \beta, \quad -\phi - \alpha \delta \omega.$$

Since none of them can be zero (to avoid zero divisors), we need to require:

$$\alpha = -\epsilon \omega, \quad \beta = -\epsilon, \quad \delta = \epsilon, \quad \phi = -1.$$

Using these constants, the determinants when two components are zero become:

$$\begin{aligned} &\epsilon^2(y_1^2 + y_3^2)(y_1^2 + \omega^2 y_3^2), \quad (y_0^2 + y_2^2)^2, \\ &y_0^4 + \epsilon^2 y_1^4, \quad y_0^4 + \epsilon^2 \omega^2 y_3^4, \quad y_2^4 + \epsilon^2 y_1^4, \quad y_2^4 + \epsilon^2 \omega^2 y_3^4. \end{aligned}$$

Since the structure constant entries are in $\{1, -1\}$, have we $\epsilon^2 = \omega^2 = 1$. There are clearly no zero divisors when having two zero components. Identical constraints are obtained from $\det(M^R)$ for two components being zero.

We proceed to consider the determinant of M^L when there is one zero component. In particular:

$$\det M^L|_{y_3=0} = y_1^4 + 2(\epsilon - 1)y_2 y_0 y_1^2 + (y_0^2 + y_2^2)^2.$$

Using the substitution $y_1^2 = u$ we solve $\det M^L = 0$ for u to obtain:

$$u = y_1^2 = (1 - \epsilon)y_0 y_2 \pm \sqrt{y_0^2 y_2^2 (\epsilon - 1)^2 - (y_0^2 + y_2^2)^2}.$$

There are thus nontrivial solutions for $\epsilon = -1$. For instance $y_0 = y_2 = 1$, $y_3 = 0$, and $y_1^2 = 2$ would produce $\det M^L = 0$. Hence, we need to require $\epsilon = 1$. We adopt this, and turn to evaluate another determinant:

$$\det M^L|_{y_2=0} = y_0^4 + 2(\omega - 1)y_3 y_1 y_0^2 + (y_1^2 + y_3^2)^2.$$

Through a quite analogous procedure we conclude that we need to require $\omega = 1$ in order to avoid zero divisors.

We obtain, thus, a single candidate for structure constant array of a not-necessarily-associative division algebra when using a normalized left-standard basis:

$C_{\mathbb{T}_L}$	0	1	2	3
0	1	1	1	1
1	1	1	1	-1
2	1	-1	-1	1
3	1	1	-1	1

Table V: Structure constant array of \mathbb{T}_L

Similarly, there is a unique choice of structure constant when a normalized right-standard basis (instead of a left-standard basis) is adopted. This is the opposite algebra to \mathbb{T}_L , denoted \mathbb{T}_R , that will turn out to be isomorphic to \mathbb{T}_L itself, and has structure constant equal to the transposed of the array $C_{\mathbb{T}_L}$ obtained above. That is $C_{\mathbb{T}_R} = (C_{\mathbb{T}_L})^{tr}$.

The use of this structure constant $C_{\mathbb{T}_L}$ leads to the algebra \mathbb{T}_L , where:

$$\begin{aligned} \det M_{\mathbb{T}_L}^L &= (y_0^2 + y_2^2)^2 + (y_1^2 + y_3^2)^2, \\ \det M_{\mathbb{T}_L}^R &= (x_0^2 + x_2^2)^2 + (x_1^2 + x_3^2)^2. \end{aligned}$$

Similar expressions are obtained for the opposite algebra \mathbb{T}_R . The absence of zero divisors follows again from the positive definiteness of the determinants. Obviously, a zero divisor from the left would imply a zero divisor from the right. The product turns out to be noncommutative (non-symmetric array $C_{\mathbb{T}_L}$). There is also a hint to understand this algebra in terms of pairs of complex numbers, say $[x_0, x_2]_{\mathbb{C}}$ and $[x_1, x_3]_{\mathbb{C}}$ that we will consider below. We call the resulting not-necessarily associative division algebra the **Tesseractian algebra** \mathbb{T}_L . The elements of this algebra will be denoted,

$$x \equiv x_0 + x_1 w + x_2 w^2 + x_3 w^3 \equiv [x_0, x_1, x_2, x_3]_{\mathbb{T}_L}. \quad (22)$$

The product of \mathbb{T}_L in terms of components takes the form:

$$\begin{aligned} x \cdot y &= [x_0, x_1, x_2, x_3]_{\mathbb{T}_L} \cdot [y_0, y_1, y_2, y_3]_{\mathbb{T}_L} \\ &= [x_0 y_0 - x_2 y_2 - x_1 y_3 + x_3 y_1, x_0 y_1 + x_2 y_3 + x_1 y_0 - x_3 y_2, \\ &\quad x_0 y_2 + x_2 y_0 + x_1 y_1 + x_3 y_3, x_0 y_3 - x_2 y_1 + x_1 y_2 + x_3 y_0]_{\mathbb{T}_L}. \end{aligned} \quad (23)$$

The expression of elements of \mathbb{T}_R is similar, and the product $y \cdot_R x$ in \mathbb{T}_R is given by the expression of the product $x \cdot y$ in \mathbb{T}_L , since \mathbb{T}_R is the opposite algebra to \mathbb{T}_L . We observe that the element $w^2 \in \mathbb{T}_L$ generates a subalgebra isomorphic to \mathbb{C} . Let us identify for this consideration $i \equiv w^2$. A generic tesseracton element $x \in \mathbb{T}_L$ can be rewritten as a pair of complex numbers:

$$\begin{aligned} x &\equiv x_0 + x_1 w + x_2 w^2 + x_3 w^3 = (x_0 + x_2 i) + w \cdot (x_1 + x_3 i) \\ &= [x_0, x_2]_{\mathbb{C}} + w \cdot [x_1, x_3]_{\mathbb{C}} \equiv ([x_0, x_2]_{\mathbb{C}}, [x_1, x_3]_{\mathbb{C}})_{\mathbb{T}_L} \\ &\equiv (X_{Even}, X_{Odd})_{\mathbb{T}_L}, \end{aligned} \quad (24)$$

$$\text{where} \quad X_{Even} \equiv [x_0, x_2]_{\mathbb{C}}, \quad X_{Odd} \equiv [x_1, x_3]_{\mathbb{C}}. \quad (25)$$

We employed above the substitution $w^3 = w \cdot w^2 = w \cdot i$. Using the structure constant in Table V, we can obtain the following properties of the tesseracton algebra:

$$\begin{aligned} (w \cdot X_{Odd}) \cdot Y_{Even} &= w \cdot (X_{Odd} \cdot Y_{Even}), \\ X_{Even} \cdot (w \cdot Y_{Odd}) &= w \cdot (Y_{Odd} \cdot \bar{X}_{Even}), \\ (w \cdot X_{Odd}) \cdot (w \cdot Y_{Odd}) &= (\bar{X}_{Odd} \cdot Y_{Odd}) \cdot w^2 = (\bar{X}_{Odd} \cdot Y_{Odd}) \cdot i. \end{aligned}$$

Employing these properties, the product in \mathbb{T}_L can be written as:

$$\begin{aligned} x \cdot y &= (X_{Even}, X_{Odd})_{\mathbb{T}_L} \cdot (Y_{Even}, Y_{Odd})_{\mathbb{T}_L} \\ &= (Y_{Even} \cdot X_{Even} + (\bar{X}_{Odd} \cdot Y_{Odd}) \cdot i, \\ &\quad X_{Odd} \cdot ((i \cdot Y_{Even}) \cdot (-i)) + Y_{Odd} \cdot ((i \cdot \bar{X}_{Even}) \cdot (-i)))_{\mathbb{T}_L}, \end{aligned} \quad (26)$$

which generalizes the Cayley-Dickson process for extensions moving from a \mathbb{Z}_a -grading to a \mathbb{Z}_{2a} -grading. We introduced there some factors i and $-i$ that can be canceled out, but they will gain relevance in further extensions (see [19]). Observe nevertheless, that the tesseracton algebra inherits a \mathbb{Z}_2 -grading since in the product of pairs $(X_{Even}, X_{Odd})_{\mathbb{T}_L}$ the contributions to the even part comes from product of two even or two odd parts, and the contribution to the odd part comes from products of even and odd parts. Observe also that the corresponding opposite algebra \mathbb{T}_R will have a product “ \cdot_R ” of the form (exchanging x and y above):

$$\begin{aligned} x \cdot_R y &= (X_{Even}, X_{Odd})_{\mathbb{T}_R} \cdot_R (Y_{Even}, Y_{Odd})_{\mathbb{T}_R} \\ &= (X_{Even} \cdot Y_{Even} + i \cdot (X_{Odd} \cdot \bar{Y}_{Odd}), \\ &\quad Y_{Odd} \cdot X_{Even} + X_{Odd} \cdot \bar{Y}_{Even})_{\mathbb{T}_R}, \end{aligned} \quad (27)$$

Accordingly, this product belongs to a family of generalized Cayley-Dickson doublings considered in [8], and [7] leading to what have been called **nonassociative**

quaternions. This family was proven to be equivalent to some 4-dimensional NNA division \mathbb{R} -algebras already considered in [6], [11], and [12]. As far as the possible underlying grading groups for 4-dimensional NNA division algebras, we find some NNA division \mathbb{R} -algebras that demand at least two algebra generators and have grading group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The name *nonassociative quaternions* for such algebras seems very natural and consonant. But some other NNA division \mathbb{R} -algebras can be generated by a single algebra element and have grading group \mathbb{Z}_4 (such as the algebra \mathbb{T}_L). From this point of view the name *nonassociative quaternions* for them would seem less natural. The question remains whether every finite dimensional division \mathbb{R} -algebra is compatible with a finite group grading for some basis choice, and more concretely, whether they are twisted group division \mathbb{R} -algebras.

We will constraint now our attention to the algebra \mathbb{T}_L that use normalized left-standard basis. The corresponding results for its opposite algebra \mathbb{T}_R that uses a normalized right-standard basis are obvious, just consider the opposite of mirror expressions in involved products. We define a conjugation in \mathbb{T}_L ,

$$\bar{x} := [x_0, -x_1, -x_2, -x_3]_{\mathbb{T}_L} = (\bar{X}_{Even}, -X_{Odd})_{\mathbb{T}_L}. \quad (28)$$

We obtain:

$$x \cdot \bar{x} = \bar{x} \cdot x = [x_0^2 + x_2^2, 0, -x_1^2 - x_3^2, 0]_{\mathbb{T}_L}.$$

Nevertheless, the product does **not** satisfy $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$ in general. Hence, \mathbb{T}_L is **not an *-algebra**. \mathbb{T}_L does not satisfy $\overline{x \cdot y} = \bar{x} \cdot \bar{y}$ in general, either. There is a rather involved relation, that allows to reproduce the effect of the conjugate in products:

$$\overline{x \cdot y} = w^3 \cdot ([w^3 \cdot (\bar{y} \cdot w)] \cdot (w^3 \cdot (\bar{x} \cdot w))) \cdot w, \quad (29)$$

$$\text{where } w = [0, 1, 0, 0]_{\mathbb{T}_L}, \text{ and } w^3 = [0, 0, 0, 1]_{\mathbb{T}_L}.$$

The product $x \cdot \bar{x}$ can be considered a complex number since from the product of \mathbb{T}_L in (26), we can identify the elements of the form $[x_0, 0, x_2, 0]_{\mathbb{T}_L} \equiv (X_{Even}, 0)_{\mathbb{T}_L}$ with the complex numbers $[x_0, x_2]_{\mathbb{C}} \equiv X_{Even}$. We define a positive definite scalar function $x \mapsto |x|_{\mathbb{T}_L} \geq 0$ with interesting properties:

$$\begin{aligned} |x|_{\mathbb{T}_L}^4 &:= \overline{(x \cdot (x \cdot \bar{x}))} \cdot x = \overline{(\bar{x} \cdot x^2)} \cdot x = x \cdot \overline{((x \cdot \bar{x}) \cdot x)} = x \cdot \overline{(x^2 \cdot \bar{x})} \quad (30) \\ &= (x \cdot \bar{x}) \cdot \overline{(x \cdot \bar{x})} = [(x_0^2 + x_2^2)^2 + (x_1^2 + x_3^2)^2, 0, 0, 0]_{\mathbb{T}_L} \\ &= |X_{Even}|_{\mathbb{C}}^4 + |X_{Odd}|_{\mathbb{C}}^4 = (x_0^2 + x_2^2)^2 + (x_1^2 + x_3^2)^2. \quad (31) \end{aligned}$$

For $x \neq 0$ and using the identities in (30), we obtain the left- and right- inverses in \mathbb{T}_L in terms of the scalar quantity and products involving conjugates:

$$LI(x) = \frac{1}{|x|_{\mathbb{T}_L}^4} \overline{(x \cdot (x \cdot \bar{x}))} = \frac{1}{|x|_{\mathbb{T}_L}^4} \overline{(\bar{x} \cdot x^2)}, \quad (32)$$

$$RI(x) = \frac{1}{|x|_{\mathbb{T}_L}^4} \overline{((x \cdot \bar{x}) \cdot x)} = \frac{1}{|x|_{\mathbb{T}_L}^4} \overline{(x^2 \cdot \bar{x})}. \quad (33)$$

The left- and right- inverses are different in general, and so the tesseraion algebra \mathbb{T}_L is a not-necessarily-associative division algebra **with chiral inverses**.

Although, for all $x \in \mathbb{T}_L$, we have $x + \bar{x} \in \mathbb{R}$, we do not have $x \cdot \bar{x}$ real, but we do have $x \cdot \bar{x} \in \mathbb{C}$, and $|x|_{\mathbb{T}_L} > 0$, for $x \neq 0$. This leads to suggest the definition of a **higher order norm**: There is a conjugate \tilde{x} that produces through $x \cdot \tilde{x}$ an element of division subalgebra of smaller dimension which has itself a higher order norm or norm. We proceed in this manner until we arrive through an even root of a monomial in x and its conjugate(s) to define a value $|x| \in \mathbb{R}$ which is an even root of the positive definite determinant M^R . In the case of \mathbb{T}_L the steps are $\mathbb{T}_L \rightarrow \mathbb{C} \rightarrow \mathbb{R}$. The number of steps would give the order of the norm. So, $|x|_{\mathbb{T}_L}$ would be a 2nd order norm.

In general, we do not need to require that the conjugation in the diverse stages of a higher order norm coincide (when we regard the conjugated objects as elements of the largest algebra), although in the case of \mathbb{T}_L they do. There remain the question if the so defined “higher order norm” satisfies the Schwarz or the triangle inequalities. We perform some short calculations. For

$$p := [1, 1, 0, 0]_{\mathbb{T}_L}, \quad q := [1, -1, 0, 0]_{\mathbb{T}_L}, \quad s := [1, 1, 1, 0]_{\mathbb{T}_L}, \quad t := [1, -1, 1, 0]_{\mathbb{T}_L},$$

we obtain

$$|p|_{\mathbb{T}_L} |p|_{\mathbb{T}_L} - |p \cdot p|_{\mathbb{T}_L} = -16, \quad |p|_{\mathbb{T}_L} |q|_{\mathbb{T}_L} - |p \cdot q|_{\mathbb{T}_L} = 0, \quad |s|_{\mathbb{T}_L} |t|_{\mathbb{T}_L} - |s \cdot t|_{\mathbb{T}_L} = 8.$$

The **Schwarz inequality** is clearly **not fulfilled**. Hence, \mathbb{T}_L is **not a normed NNA division \mathbb{R} -algebra**. Nevertheless, we find for the same elements of \mathbb{T}_L :

$$\begin{aligned} |p|_{\mathbb{T}_L} + |p|_{\mathbb{T}_L} - |p + p|_{\mathbb{T}_L} &= 0, \quad |p|_{\mathbb{T}_L} + |q|_{\mathbb{T}_L} - |p + q|_{\mathbb{T}_L} = 2(2^{1/4} - 1) > 0, \\ |s|_{\mathbb{T}_L} + |t|_{\mathbb{T}_L} - |s + t|_{\mathbb{T}_L} &= 2(5^{1/4} - 2^{1/2}) > 0. \end{aligned}$$

The validity of the triangle inequality is a particular case of the family of norms proved in Appendix A. Hence, the function $x \mapsto |x|_{\mathbb{T}_L}$ **is actually a norm**. Hence, although \mathbb{T}_L is not a normed division algebra (since its norm does not satisfy the Schwarz equality), it does have a norm. There is also remarkable property, which constitutes a generalization of positive homogeneity (13), where the scalar is taken from a proper division subalgebra.

Definition 10. We call **pure even elements** of \mathbb{T}_L those $x \in \mathbb{T}_L$ such that $x_1 = x_3 = 0$. Similarly we define **pure odd elements**.

$$x_1 = x_3 = 0 \iff x \text{ is pure even element}, \quad (34)$$

$$x_0 = x_2 = 0 \iff x \text{ is pure odd element}. \quad (35)$$

Although the Schwarz inequality is not satisfied in general, when any of the two factors is pure even or pure odd then the property holds with strict equality:

$$X_{Even} = [x_0, x_2]_{\mathbb{C}} = 0 \implies |x \cdot y|_{\mathbb{T}_L} = |x|_{\mathbb{T}_L} |y|_{\mathbb{T}_L}, \quad (36)$$

$$X_{Odd} = [x_1, x_3]_{\mathbb{C}} = 0 \implies |x \cdot y|_{\mathbb{T}_L} = |x|_{\mathbb{T}_L} |y|_{\mathbb{T}_L},$$

$$Y_{Even} = [y_0, y_2]_{\mathbb{C}} = 0 \implies |x \cdot y|_{\mathbb{T}_L} = |x|_{\mathbb{T}_L} |y|_{\mathbb{T}_L},$$

$$Y_{Odd} = [y_0, y_2]_{\mathbb{C}} = 0 \implies |x \cdot y|_{\mathbb{T}_L} = |x|_{\mathbb{T}_L} |y|_{\mathbb{T}_L}, \quad (37)$$

for all $x, y \in \mathbb{T}_L$.

It turns out that pure even elements are universally associative (but not universally commutative), that is:

$$x_1 = x_3 = 0 \implies x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad (38)$$

$$x_1 = x_3 = 0 \implies y \cdot (x \cdot z) = (y \cdot x) \cdot z,$$

$$x_1 = x_3 = 0 \implies y \cdot (z \cdot x) = (y \cdot z) \cdot x, \quad (39)$$

for all $x, y, x \in \mathbb{T}_L$.

This property is also called \mathbb{C} -associativity since even elements constitute a sub-algebra isomorphic to \mathbb{C} which is in the nucleus of the algebra. This establishes also a \mathbb{C} -bimodule structure of \mathbb{T}_L : a left (or right) \mathbb{C} -module using left- (or right-) multiplication by pure even factors (which as said, are universally associative). There is also an action of the form $z \mapsto x \cdot (z \cdot x^{-1})$, for each $x \neq 0$ pure even, since pure even are universally associative. The study of orbits of elements $z \in \mathbb{T}_L$ with $|z|_{\mathbb{T}_L} = 1$ under left- or right-multiplication by pure even or pure odd elements $x \in \mathbb{T}_L$ with $|x|_{\mathbb{T}_L} = 1$ leads to a rich characterization of a unit tesseract manifold and considerations analogous to the Hopf fibrations using quaternions and octonions (see [28], [29], [41]). On the other hand the relations (38-39) reveal that if we express each $x \in \mathbb{T}_L$ as a sum of pure even and pure odd parts, only expressions in which each of the three factors are pure odd parts contribute to the non-associativity. The associator turns out to be a cubic expression of odd parts, which is in turn also odd. As we will see, any non zero odd element (with or without unit norm) generates the algebra. Also, we find $x \in \mathbb{T}_L$ with non zero odd part satisfies $x \cdot x^2 - x^2 \cdot x \neq 0$. Hence, it generates the whole algebra.

The tesseract algebra \mathbb{T}_L is **not power associative nor flexible**, since,

$$w \cdot (w \cdot w) = -(w \cdot w) \cdot w, \quad \text{for } w = [0, 1, 0, 0]_{\mathbb{T}_L}.$$

Hence, it is **not alternative** either in consonance with the famous Zorn theorem about alternative division \mathbb{R} -algebras. Explicit computation using the \mathbb{T}_L -algebra product leads to verify that \mathbb{T}_L does not fulfill the left- nor the right- Bol identity. It does not fulfill the Moufang identities. Hence, the algebra \mathbb{T}_L is **neither a left- nor a right-Bol loop**, and it is **not a Moufang loop** either.

The question is now, which are the identities that characterize the tesseractians? We want to devote some attention to some of them since they might be used to tell apart diverse extensions of the tesseractians to be studied in this paper series. The equations (30) give a hint of some of the identities. They follow from the following two cubic identities that hold in \mathbb{T}_L that we call **cubic tesseractianity identity involving conjugates identity in two variables**:

$$\bar{x} \cdot (x \cdot y) = y \cdot (x \cdot \bar{x}), \quad (40)$$

$$(y \cdot x) \cdot \bar{x} = (x \cdot \bar{x}) \cdot y, \quad \text{for all } x, y \in \mathbb{T}_L. \quad (41)$$

These identities allow for solving the linear equations $a \cdot x = c$ and $y \cdot b = d$ for the unknowns $x, y \in \mathbb{T}_L$ when a, b are non zero:

$$a \cdot x = c \implies x = \frac{1}{|a|_{\mathbb{T}_L}^4} (\bar{a} \cdot c) \cdot \overline{(a \cdot \bar{a})}, \quad \forall x, c, a \in \mathbb{T}_L, \quad a \neq 0; \quad (42)$$

$$y \cdot b = d \implies y = \frac{1}{|b|_{\mathbb{T}_L}^4} (\overline{b \cdot \bar{b}}) \cdot (d \cdot \bar{b}), \quad \forall y, d, b \in \mathbb{T}_L, \quad b \neq 0. \quad (43)$$

We used there the fact that $a \cdot \bar{a}$ and $b \cdot \bar{b}$ are pure even, and thus universally associative. These formulas provide a **method of encryption**. For instance, taking all components in \mathbb{Z}_p , for p prime, and $|a|_{\mathbb{T}_L} \neq 0 \pmod p$, we can codify c through x (encoded word) as in (42), which can be easily decoded multiplying by a , since $c = a \cdot x$. Observe that information theory applications of such algebras have been already started in [32].

We study all generic cubic polynomial identities in two variables and obtain one single identity that we call the **cubic tesseract identity in two variables** (not involving conjugates):

$$[x \cdot (x \cdot y) - x^2 \cdot y] - [y \cdot x^2 - (y \cdot x) \cdot x] = 0, \quad \forall x, y \in \mathbb{T}_L. \quad (44)$$

Observe that the identity above is a linear combination of the two identities satisfied by alternative algebras. We study all generic quartic polynomial identities in one variable satisfied by the tesseract algebra \mathbb{T}_L , and obtain two independent **quartic tesseract identities in a single variable** (the first one follows from (44) taking $y = x^2$):

$$[x \cdot (x \cdot x^2) - x^2 \cdot x^2] - [x^2 \cdot x^2 - (x^2 \cdot x) \cdot x] = 0, \quad (45)$$

$$[x \cdot (x^2 \cdot x) - (x \cdot x^2) \cdot x] - [x^2 \cdot x^2 - (x^2 \cdot x) \cdot x] = 0, \quad \forall x \in \mathbb{T}_L. \quad (46)$$

We study now generic quartic polynomial identities in two variables quadratic in each of them:

$$\begin{aligned} & a_1 (x \cdot x) \cdot (y \cdot y) + a_2 (x \cdot y) \cdot (x \cdot y) + a_3 (y \cdot x) \cdot (x \cdot y) \\ & + a_4 (x \cdot y) \cdot (y \cdot x) + a_5 (y \cdot x) \cdot (y \cdot x) + a_6 (y \cdot y) \cdot (x \cdot x) \\ & + b_1 x \cdot (x \cdot (y \cdot y)) + b_2 x \cdot (y \cdot (x \cdot y)) + b_3 y \cdot (x \cdot (x \cdot y)) \\ & + b_4 x \cdot (y \cdot (y \cdot x)) + b_5 y \cdot (x \cdot (y \cdot x)) + b_6 y \cdot (y \cdot (x \cdot x)) \\ & + c_1 ((x \cdot x) \cdot y) \cdot y + c_2 ((x \cdot y) \cdot x) \cdot y + c_3 ((y \cdot x) \cdot x) \cdot y \\ & + c_4 ((x \cdot y) \cdot y) \cdot x + c_5 ((y \cdot x) \cdot y) \cdot x + c_6 ((y \cdot y) \cdot x) \cdot x \\ & + e_1 x \cdot ((x \cdot y) \cdot y) + e_2 x \cdot ((y \cdot x) \cdot y) + e_3 y \cdot ((x \cdot x) \cdot y) \\ & + e_4 x \cdot ((y \cdot y) \cdot x) + e_5 y \cdot ((x \cdot y) \cdot x) + e_6 y \cdot ((y \cdot x) \cdot x) \\ & + f_1 (x \cdot (x \cdot y)) \cdot y + f_2 (x \cdot (y \cdot x)) \cdot y + f_3 (y \cdot (x \cdot x)) \cdot y \\ & + f_4 (x \cdot (y \cdot y)) \cdot x + f_5 (y \cdot (x \cdot y)) \cdot x + f_6 (y \cdot (y \cdot x)) \cdot x = 0. \end{aligned} \quad (47)$$

We test the identity with diverse values $x, y \in \mathbb{T}_L$ and then obtain the necessary relations that lead to an identity for arbitrary x, y values:

$$\begin{aligned}
a_1 &= -f_6 - c_1 - c_6 - f_2 + f_3, & a_2 &= -c_4 - f_4 - c_2, & a_3 &= -f_3 - c_2 - c_3, \\
a_4 &= -c_5 - f_4 - c_4, & a_5 &= -f_3 - c_3 - c_5, & a_6 &= -c_1 - f_1 - f_5 + f_4 - c_6, \\
b_1 &= 2f_6 + c_6 + 2f_2 + e_4 + f_4 - f_1 - f_5 - f_3, & b_2 &= c_4 + f_4, \\
b_3 &= c_2 + f_1 + f_5 + f_3 - f_2 + e_6, & b_4 &= -e_4 + c_5, & b_5 &= f_3 + c_3, \\
b_6 &= -f_4 - f_6 - e_6 + f_1 + f_5 + c_1, & e_1 &= -f_6 + f_5 - f_2 - e_4 - f_4, \\
e_2 &= -f_2, & e_3 &= -f_1 - f_5 - f_3 + f_2 - e_6, & e_5 &= -f_5.
\end{aligned} \tag{48}$$

Therefore, there are 14 independent coefficients leading to 14 identities. Four of the resulting identities can be obtained from the others interchanging the roles of x and y . We find then that all **quartic tesseranities in two variables quadratic in each of them** are a linear combination of the following 10 identities (and those obtained exchanging the two variables):

$$[x^2 \cdot y^2 - (x^2 \cdot y) \cdot y] - [y \cdot (y \cdot x^2) - y^2 \cdot x^2] = 0, \tag{49}$$

$$y \cdot ([x \cdot (x \cdot y) - x^2 \cdot y] - [y \cdot x^2 - (y \cdot x) \cdot x]) = 0, \tag{50}$$

$$[(x \cdot y)^2 - ((x \cdot y) \cdot x) \cdot y] - [y \cdot (x \cdot (x \cdot y)) - (y \cdot x) \cdot (x \cdot y)] = 0, \tag{51}$$

$$[x \cdot (y \cdot (x \cdot y)) - (x \cdot y)^2] - [(x \cdot y) \cdot (y \cdot x) - ((x \cdot y) \cdot y) \cdot x] = 0, \tag{52}$$

$$\begin{aligned}
&x \cdot [x \cdot y^2 - (x \cdot y) \cdot y] + y \cdot [x^2 \cdot y - x \cdot (x \cdot y)] \\
&+ [x \cdot ((x \cdot y) \cdot y) - (x \cdot (x \cdot y)) \cdot y] - [y \cdot (y \cdot x^2) - y^2 \cdot x^2] = 0,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&x \cdot [x \cdot y^2 - (x \cdot y) \cdot y] + y \cdot [x^2 \cdot y - x \cdot (x \cdot y)] \\
&+ [x \cdot (x \cdot y^2) - x^2 \cdot y^2] - [x \cdot ((y \cdot x) \cdot y) - (x \cdot (y \cdot x)) \cdot y] = 0,
\end{aligned} \tag{54}$$

$$\begin{aligned}
&[x \cdot (x \cdot y^2) - x^2 \cdot y^2] - [y \cdot (x \cdot (y \cdot x)) - (y \cdot x)^2] \\
&+ [y \cdot (x^2 \cdot y) - (y \cdot x^2) \cdot y] - [y \cdot (x \cdot (x \cdot y)) - (y \cdot x) \cdot (x \cdot y)] = 0,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&x \cdot [(x \cdot y) \cdot y - x \cdot y^2] + [(x \cdot y) \cdot y - x \cdot y^2] \cdot x \\
&+ [y \cdot (y \cdot x^2) - y^2 \cdot x^2] = 0,
\end{aligned} \tag{56}$$

$$[y \cdot ((x \cdot y) \cdot x) - (y \cdot (x \cdot y)) \cdot x] - [x \cdot ((x \cdot y) \cdot y) - (x \cdot (x \cdot y)) \cdot y] = 0, \tag{57}$$

$$\begin{aligned}
&x \cdot [x \cdot y^2 - (x \cdot y) \cdot y] + y \cdot [(y \cdot x) \cdot x - y \cdot x^2] \\
&+ [x \cdot (x \cdot y^2) - x^2 \cdot y^2] - [y \cdot ((y \cdot x) \cdot x) - (y \cdot (y \cdot x)) \cdot x] = 0,
\end{aligned} \tag{58}$$

$$\forall x, y \in \mathbb{T}_L.$$

These identities seem to suggest that perhaps some few cubic and quartic associator identities would generate all of them. In fact, the first of them in (49) is obtained from the cubic identity in two variables in (44) by first replacing y by y^2 , and then exchanging the x and y variables. The identity in (50) is just the identity in (44) multiplied by y from the left.

We will find in these paper series, see [19], algebras that satisfy the cubic identity in two variables in (44) as well as the quartic identities in (49–50), which do not satisfy any of the other quartic identities in two variables (51–58). Hence, they can not follow just from the cubic identity in two variables.

We obtain also **quartic identities which are cubic in one variable and linear in the other**:

$$\begin{aligned}
& a_1 x^2 \cdot (x \cdot y) + a_2 x^2 \cdot (y \cdot x) + a_3 (x \cdot y) \cdot x^2 + a_4 (y \cdot x) \cdot x^2 \\
& + b_1 x \cdot (x \cdot (x \cdot y)) + b_2 x \cdot (x \cdot (y \cdot x)) + b_3 x \cdot (y \cdot x^2) + b_4 y \cdot (x \cdot x^2) \\
& + c_1 (x^2 \cdot x) \cdot y + c_2 (x^2 \cdot y) \cdot x + c_3 ((x \cdot y) \cdot x) \cdot x + c_4 ((y \cdot x) \cdot x) \cdot x \quad (59) \\
& + e_1 x \cdot (x^2 \cdot y) + e_2 x \cdot ((x \cdot y) \cdot x) + e_3 x \cdot ((y \cdot x) \cdot x) + e_4 y \cdot (x^2 \cdot x) = 0,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= -c_3 - c_1 - f_3, \quad a_2 = -c_4 - f_3 - c_2, \quad a_3 = -c_3 - f_2 - c_2, \\
a_4 &= -f_1 - f_4 + f_3 - c_1 - c_4, \quad b_1 = c_3 + e_3 + 2f_3, \quad b_2 = c_4 + 2f_3 - f_2, \\
b_3 &= f_2 - f_3 + c_2 - e_3, \quad b_4 = f_1 + f_4 - 2f_3 + c_1 + f_2, \quad e_1 = -f_1 - e_3 - f_3, \\
e_2 &= -f_3, \quad e_4 = -f_2 - f_4 + f_3. \quad (60)
\end{aligned}$$

There are thus nine independent coefficients leading to nine quartic identities cubic in one variable and linear in the other.

The obtained quartic identities complement the quartic identities presented in (30), that we call **quartic tesseranity identities involving conjugates in one variable** (where the last equality in each line follows from equations (40-41)):

$$(x \cdot \bar{x}) \cdot \overline{(x \cdot \bar{x})} = \overline{(x \cdot (x \cdot \bar{x}))} \cdot x = \overline{(\bar{x} \cdot x^2)} \cdot x \quad (61)$$

$$= x \cdot \overline{((x \cdot \bar{x}) \cdot x)} = x \cdot \overline{(x^2 \cdot \bar{x})}, \quad \forall x \in \mathbb{T}_L. \quad (62)$$

We obtained also the **quintic identities in one variable**. There are as many quintic monomial summands as the fourth catalan number $c(4) = 14$:

$$\begin{aligned}
& a_1 x \cdot (x^2 \cdot x^2) + b_1 x \cdot (x \cdot (x \cdot x^2)) + c_1 x \cdot ((x^2 \cdot x) \cdot x) + e_1 x \cdot (x \cdot (x^2 \cdot x)) \\
& + f_1 x \cdot ((x \cdot x^2) \cdot x) + a_2 (x^2 \cdot x^2) \cdot x + b_2 (x \cdot (x \cdot x^2)) \cdot x \\
& + c_2 ((x^2 \cdot x) \cdot x) \cdot x + e_2 (x \cdot (x^2 \cdot x)) \cdot x + f_2 ((x \cdot x^2) \cdot x) \cdot x \\
& + a_3 x^2 \cdot (x^2 \cdot x) + b_3 x^2 \cdot (x \cdot x^2) + c_3 (x^2 \cdot x) \cdot x^2 + e_3 (x \cdot x^2) \cdot x^2 = 0, \quad (63)
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= -2c_1 - 2f_1 - b_2 + c_2 - 2e_2 - e_1 - f_2 - e_3, \\
a_2 &= -2c_2 + f_1 + e_1 - f_2 - e_3 - c_3, \\
a_3 &= -f_1 - b_2 + c_2 - e_1 + f_2 + e_3 + c_3, \\
b_1 &= c_1 + 2f_1 + b_2 - c_2 + 2e_2 + e_1 + f_2, \\
b_3 &= -e_1 - e_2 - f_1 - f_2 - c_3. \quad (64)
\end{aligned}$$

There are nine independent coefficient and thus nine quintic identities in one variable satisfied by the algebra \mathbb{T}_L .

We finish showing the **identities of order six identities in one variable** satisfied by the tessaranion algebra:

$$\begin{aligned}
& a_{01} x \cdot (x \cdot (x \cdot (x \cdot x^2))) + a_{02} x \cdot (x \cdot (x \cdot (x^2 \cdot x))) \\
& + a_{03} x \cdot (x \cdot (x^2 \cdot x^2)) + a_{04} x \cdot (x \cdot ((x \cdot x^2) \cdot x)) \\
& + a_{05} x \cdot (x \cdot ((x^2 \cdot x) \cdot x)) + a_{06} x \cdot (x^2 \cdot (x \cdot x^2)) \\
& + a_{07} x \cdot (x^2 \cdot (x^2 \cdot x)) + a_{08} x \cdot ((x \cdot x^2) \cdot x^2) \\
& + a_{09} x \cdot ((x \cdot (x \cdot x^2)) \cdot x) + a_{10} x \cdot ((x \cdot (x^2 \cdot x)) \cdot x) \\
& + a_{11} x \cdot ((x^2 \cdot x) \cdot x^2) + a_{12} x \cdot ((x^2 \cdot x^2) \cdot x) \\
& + a_{13} x \cdot (((x \cdot x^2) \cdot x) \cdot x) + a_{14} x \cdot (((x^2 \cdot x) \cdot x) \cdot x) \\
& + a_{15} x^2 \cdot (x \cdot (x \cdot x^2)) + a_{16} x^2 \cdot (x \cdot (x^2 \cdot x)) \\
& + a_{17} x^2 \cdot (x^2 \cdot x^2) + a_{18} x^2 \cdot ((x \cdot x^2) \cdot x) \\
& + a_{19} x^2 \cdot ((x^2 \cdot x) \cdot x) + a_{20} (x \cdot x^2) \cdot (x \cdot x^2) \\
& + a_{21} (x \cdot x^2) \cdot (x^2 \cdot x) + a_{22} (x \cdot (x \cdot x^2)) \cdot x^2 \\
& + a_{23} (x \cdot (x \cdot (x \cdot x^2))) \cdot x + a_{24} (x \cdot (x \cdot (x^2 \cdot x))) \cdot x \\
& + a_{25} (x \cdot (x^2 \cdot x)) \cdot x^2 + a_{26} (x \cdot (x^2 \cdot x^2)) \cdot x \\
& + a_{27} (x \cdot ((x \cdot x^2) \cdot x)) \cdot x + a_{28} (x \cdot ((x^2 \cdot x) \cdot x)) \cdot x \\
& + a_{29} (x^2 \cdot x) \cdot (x \cdot x^2) + a_{30} (x^2 \cdot x) \cdot (x^2 \cdot x) \\
& + a_{31} (x^2 \cdot x^2) \cdot x^2 + a_{32} (x^2 \cdot (x \cdot x^2)) \cdot x \\
& + a_{33} (x^2 \cdot (x^2 \cdot x)) \cdot x + a_{34} ((x \cdot x^2) \cdot x) \cdot x^2 \\
& + a_{35} ((x \cdot x^2) \cdot x^2) \cdot x + a_{36} ((x \cdot (x \cdot x^2)) \cdot x) \cdot x \\
& + a_{37} ((x \cdot (x^2 \cdot x)) \cdot x) \cdot x + a_{38} ((x^2 \cdot x) \cdot x) \cdot x^2 \\
& + a_{39} ((x^2 \cdot x) \cdot x^2) \cdot x + a_{40} ((x^2 \cdot x^2) \cdot x) \cdot x \\
& + a_{41} (((x \cdot x^2) \cdot x) \cdot x) \cdot x + a_{42} (((x^2 \cdot x) \cdot x) \cdot x) \cdot x = 0, \quad (65)
\end{aligned}$$

where

$$\begin{aligned}
a_{01} &= 2a_{36} + a_{31} + a_{33} + a_{29} + 2a_{34} + 2a_{28} + a_{20} + a_{35} + a_{40} + a_{38} \\
&+ 2a_{37} + a_{41} + 2a_{25} + a_{24} + 2a_{10} + a_{14} + a_{26} + a_{12} + a_{09} + 2a_{27} \\
&+ a_{04} + a_{05} + a_{11} + 2a_{13} + a_{08} + a_{22}, \\
a_{02} &= 2a_{36} + 2a_{31} + a_{33} + a_{29} + 2a_{34} + a_{30} + 2a_{28} + a_{21} + a_{32} + a_{20} \\
&+ 2a_{35} + 2a_{40} + 2a_{38} + 3a_{37} + 3a_{41} + 2a_{25} + a_{24} + 2a_{42} + 2a_{39} \\
&+ 2a_{14} + a_{26} + a_{12} + 2a_{27} - a_{04} + a_{11} + a_{13} + a_{08} + 2a_{22}, \\
a_{03} &= -a_{36} + a_{30} - a_{28} + a_{21} + a_{32} + a_{35} + a_{40} + a_{38} - a_{37} + a_{41} - a_{25} \\
&- a_{24} + a_{23} - 2a_{10} + 3a_{42} + 2a_{39} - a_{14} - a_{12} - a_{09} - 2a_{27} - a_{04} \\
&- 2a_{05} - a_{11} - 2a_{13} - 2a_{08} - a_{22}, \quad (66)
\end{aligned}$$

and

$$\begin{aligned}
a_{06} &= -2a_{36} - 2a_{31} - a_{33} - a_{29} - 3a_{34} - a_{30} - 2a_{28} - a_{21} - a_{32} - 2a_{20} \\
&\quad - 2a_{35} - 2a_{40} - 2a_{38} - 3a_{37} - 3a_{41} - 3a_{25} - a_{24} - a_{10} - 2a_{42} - 2a_{39} \\
&\quad - 2a_{14} - a_{26} - a_{12} - 2a_{27} - 2a_{11} - 2a_{13} - a_{08} - 2a_{22}, \\
a_{07} &= -a_{28} - a_{21} - a_{35} - a_{23} - a_{14} - a_{26} - a_{12} - a_{09}, \\
a_{15} &= -a_{41} + a_{24} + a_{19} - a_{25} - a_{28} + a_{23} - a_{31} - a_{40} - a_{35} - 2a_{29} + a_{18} \\
&\quad - a_{20} + a_{32} - 2a_{34} - a_{37} - 2a_{38} - a_{36} - a_{42}, \\
a_{16} &= -a_{30} - a_{32} - a_{37} - a_{41} - a_{24} - a_{18} - a_{39} - a_{27}, \\
a_{17} &= -a_{36} - a_{31} - 2a_{33} - a_{30} - a_{21} - 2a_{32} - a_{35} - 2a_{40} - a_{38} - a_{41} \\
&\quad - a_{24} - 2a_{23} - a_{18} - 3a_{42} - 2a_{39} - a_{26} - 2a_{19} - a_{22}. \tag{67}
\end{aligned}$$

There are thus 34 independent coefficients and 34 identities of order six in one variable satisfied by the algebra \mathbb{T}_L .

Clearly, further identities of orders 2, 3, 4, 5, 6 in several variables satisfied by the tesseract algebra can be obtained, beyond those arising from the presented above using polarization. Clearly, the identities satisfied by \mathbb{T}_R (the opposite algebra of \mathbb{T}_L) are obtained by the corresponding opposite or mirror images of the involved products. A thorough survey of polynomial identities for these and other NNA division algebras is presented in [41]. We emphasize again that we include these identities here as a tool to tell apart extensions, since some identities (or linear combination of them) hold and some don't, depending on the extension.

In terms of generators and relations, the algebra \mathbb{T}_L can be defined by

$$\begin{aligned}
< v_1 \equiv 1, v_1 \equiv w, v_2 \equiv w^2, v_3 \equiv w^3 \mid & \quad w \cdot w^2 = -w^2 \cdot w = w^3, \\
& \quad w^2 \cdot w^2 = w \cdot w^3 = -w^3 \cdot w = -1, \\
& \quad w^2 \cdot w^3 = -w^3 \cdot w^2 = w, \\
& \quad w^3 \cdot w^3 = w^2 >. \tag{68}
\end{aligned}$$

Since \mathbb{Z}_4 is abelian we can define the q - as well as the r -function. We can find that the structure constant, the q - and the r -functions are given by:

$$C_{\mathbb{T}_L}(n, m) := \exp\{i\pi[(-2n^2 + 3n - 2m^2 + m - 3nm + 3)nm/4]\}, \tag{69}$$

$$q_{\mathbb{T}_L}(n, m) := \exp\{i\pi[(n^2m - nm^2)/2]\}, \tag{70}$$

$$k_{\mathbb{T}_L}(n) := \exp\{i\pi[n^3/2 + n^2/2]\}, \quad q_{\mathbb{T}_L} = \delta k_{\mathbb{T}_L}, \tag{71}$$

$$r_{\mathbb{T}_L}(n, m, h) := \exp\{i\pi[nmh]\}, \quad \text{for all } n, m, h \in \mathbb{Z}_4. \tag{72}$$

We find once again that the algebra \mathbb{T}_L is **non-commutative and non-associative**. We find that its q -function is a **2-coboundary**, but it is **not separable**, since it does not satisfy the Jacobi-like identity (9).

We explore some **algebra automorphisms** of the algebra \mathbb{T}_L . In particular, we can ask for basis changes leading to algebras with different structure constants but isomorphic to \mathbb{T}_L . We constrain furthermore the scope to basis

changes that use a normalized left-standard basis and lead to structure constants that are unital structure constants of \mathbb{Z}_4 in $\{1, -1\}$, since our scope is to classify the twisted group algebras $(\mathcal{A}; \mathbb{Z}_4, \mathbb{R}, \{1, -1\}, C)$. We constraint thus the search to automorphisms to algebras \mathbb{T}_L generated by a generic element $w' = [a_0, a_1, a_2, a_3]_{\mathbb{T}_L}$, and with basis as vector space over \mathbb{R} given by the normalized left-standard basis $[1, w', w'^2, w' \cdot w'^2]$ (and thus generating a 4-dimensional \mathbb{R} -algebra). The generic element w' satisfies the necessary condition for the \mathbb{Z}_2 -graded subalgebra: $w'^2 \cdot w'^2 = -1$ (which makes the basis normalized). This later condition leads to $a_0 = a_2 = 0$ and $a_1^2 + a_3^2 = 1$. Using the generator $w' = [0, a_1, 0, a_3]_{\mathbb{T}_L}$, with $a_1^2 + a_3^2 = 1$, it turns out that we always obtain exactly the same structure constant $C_{\mathbb{T}_L}$ as the one obtained originally for \mathbb{T}_L . This is a rather obvious outcome since there was only a unique unital structure constant in the left-standard basis leading to a not-necessarily-associative division algebra. This implies in particular, that the algebra \mathbb{T}_L is **\mathbb{R} -algebra isomorphic to its opposite algebra \mathbb{T}_R** by a basis change. A basis for the opposite algebra \mathbb{T}_R will be $[1, w, w^2, -w \cdot w^2] = [1, w, w^2, w^2 \cdot w]$, which is thus a normalized right-standard basis, and leads to a structure constant array $C_{\mathbb{T}_R}$ which is the transpose of $C_{\mathbb{T}_L}$.

Notice, that these diverse choices of the generator w' lead to diverse graded algebras. Equivalently, the algebra \mathbb{T}_L is compatible with diverse gradings, that is, diverse partitions as a vector space into subspaces of pure degree. Notice also that we just explored some plain algebra automorphisms which are not necessarily **graded algebra automorphisms**. Notice also, that any pure odd nonzero element of \mathbb{T}_L generates the whole algebra.

7 Classification result

We explore first the non-isomorphism between the tesseronian algebra \mathbb{T}_L and the quaternion algebra \mathbb{H} . If we attempt to construct an isomorphism between \mathbb{T}_L and \mathbb{H} , we need an element $w \in \mathbb{H}$ that generates \mathbb{T}_L . Such w should satisfy $w \cdot w^2 = -w^2 \cdot w$. Nevertheless, \mathbb{H} is alternative. All elements $v \in \mathbb{H}$ satisfy $v \cdot v^2 = v^2 \cdot v$. So, \mathbb{T}_L can not be generated by an element $v \in \mathbb{H}$. Hence, **the algebras \mathbb{T}_L and \mathbb{H} are not isomorphic**. We have just proved:

Proposition 3. *The algebra of quaternions \mathbb{H} and the algebra of tesseronians \mathbb{T}_L are not isomorphic.*

We reunite all the findings above in the following classification result:

Theorem 1.

- *The twisted group algebras $(\mathcal{A}; G, K, A, C) = (\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ which are not-necessarily-associative division algebras over \mathbb{R} and have grading group of order $|G| \leq 4$ are isomorphic to one of the following mutually non-isomorphic algebras: The real algebra \mathbb{R} (a field), the complex algebra \mathbb{C} (a field), the quaternion algebra \mathbb{H} (a plain division algebra), or the*

tesseract algebra \mathbb{T}_L (a not-necessarily-associative division algebra with chiral inverses).

- The quaternion algebra as a twisted group algebra $(\mathbb{H}; \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{R}, \{1, -1\}, C_{\mathbb{H}})$ with any normalized right-standard basis $[v_{(0,0)} \equiv 1, v_{(1,0)}, v_{(0,1)}, v_{(1,1)} \equiv v_{(1,0)} \cdot v_{(0,1)}]$ has a unique structure constant $C_{\mathbb{H}}$ given in Table III. It is isomorphic to its opposite algebra, which for a left-standard algebra basis $[v_{(0,0)} \equiv 1, v_{(1,0)}, v_{(0,1)}, v_{(1,1)} \equiv v_{(0,1)} \cdot v_{(1,0)}]$ has a unique structure constant which is the transpose of $C_{\mathbb{H}}$. The structure constant-, q -, and r -functions of the quaternions are given in (15-18), where the q -function is a 2-coboundary, see (17), and it is separable.
- The tesseract algebra as a twisted group algebra $(\mathbb{T}_L; \mathbb{Z}_4, \mathbb{R}, \{1, -1\}, C_{\mathbb{T}_L})$ with any normalized left-standard basis $[v_0 \equiv 1, v_1 \equiv w, v_2 \equiv w \cdot w \equiv w^2, v_3 \equiv w \cdot (w^2) \equiv w^3]$ has a unique structure constant $C_{\mathbb{T}_L}$ given in Table V. The algebra \mathbb{T}_L is isomorphic to its opposite algebra \mathbb{T}_R . For a normalized right-standard basis $[1, w, w^2, w^2 \cdot w]$, the opposite algebra $(\mathbb{T}_R; \mathbb{Z}_4, \mathbb{R}, \{1, -1\}, C_{\mathbb{T}_R})$ has a unique structure constant array $C_{\mathbb{T}_R}$ which is the transpose of $C_{\mathbb{T}_L}$. As an algebra \mathbb{T}_L can be generated by a single element $v_1 \equiv w$. The tesseract algebra \mathbb{T}_L is non-commutative and non-associative. Its product is given in components in (23), and in terms of complex pairs in (25-26). The product of the opposite algebra \mathbb{T}_R in terms of complex pairs is given in (27). From such pairings it is evident that the complex subalgebra \mathbb{C} can be identified with the pure even (34) elements subalgebra, and the real subalgebra \mathbb{R} with tesseractians x such that $x_1 = x_2 = x_3 = 0$. The algebra \mathbb{T}_L has a conjugation from \mathbb{T}_L to \mathbb{T}_L : $x \mapsto \bar{x}$ given in (28), such that $x + \bar{x} \in \mathbb{R}$ and $x \cdot \bar{x} = \bar{x} \cdot x \in \mathbb{C}$. The conjugation is an involution but it is not an anti-homomorphism. So, \mathbb{T}_L is not an $*$ -algebra. The conjugate of a product satisfies (29). We can define a map from \mathbb{T}_L to \mathbb{R} : $x \mapsto |x|_{\mathbb{T}_L}$ which is a quartic root of a quartic monomial product of x and its conjugate \bar{x} given in (30), which is a norm, as it is proved in Appendix A in (107). The left- and the right- inverse for non-zero elements can be written in terms of a cubic monomial divided by a quartic power of the norm, as in (32-33). This positive definite norm does not satisfy the Schwarz inequality, but it satisfies the strict Schwarz equality when one of the factors is pure-even- or pure-odd- tesseractian, as in (36-37). The pure-even- tesseractians are universally associative – that is, \mathbb{T}_L is \mathbb{C} -associative-, see (38-39). This leads to a \mathbb{C} -bimodule structure: \mathbb{T}_L is a left (or right) \mathbb{C} -module using multiplication from the left (right) by pure even factors (which as said, are universally associative). There is also an action of the form $z \mapsto x \cdot (z \cdot x^{-1})$, for each x pure even. The tesseract algebra \mathbb{T}_L is not power associative, and thus not alternative. It is neither a left-Bol-, nor a right-Bol-, nor a Moufang-loop. The algebra \mathbb{T}_L satisfies the cubic identities in (40-41) and (44), and the quartic identities given in (45-62). The algebra \mathbb{T}_L satisfies also quintic and order six identities given in (63-64) and in (65-67) respectively. The identities satisfied by \mathbb{T}_R (the opposite algebra to \mathbb{T}_L) are obtained

easily by considering the opposite of mirror expressions in the products involved in the identities. In terms of generators and relations \mathbb{T}_L can be given by (68). The whole algebra \mathbb{T}_L can be generated by a single algebra element with nonzero odd part. The structure constant-, the q - and the r -functions of \mathbb{T}_L are given in (69-72). Its q -function is a 2-coboundary, but it is not separable.

8 Commutator and anti-commutator algebra of Tesseranions

We consider in this section the algebras obtained by defining two different products that use the tesseranion product:

$$[\cdot, \cdot] : \mathbb{T}_L \rightarrow \mathbb{T}_L, \quad (x, y) \mapsto [x, y] \equiv (x \cdot y - y \cdot x)/2, \quad (73)$$

$$\bullet : \mathbb{T}_L \rightarrow \mathbb{T}_L, \quad (x, y) \mapsto x \bullet y \equiv (x \cdot y + y \cdot x)/2. \quad (74)$$

The tesseranion set with its summation and the **antisymmetric or commutator product** “ $[\cdot, \cdot]$ ” constitutes the algebra \mathbb{T}_L^- or $(\mathbb{T}_L^-; +, [\cdot, \cdot])$. As a vector space, the algebra \mathbb{T}_L^- is generated by $\{v_0, v_1, v_2, v_3\}$ (and as such coincides with \mathbb{T}_L). The product can be given in terms of the structure constants $C_{\mathbb{T}_L}^-$:

$$[v_i, v_j] = C_{\mathbb{T}_L}^-(i, j) v_{(i+j) \bmod 4}, \quad (75)$$

$$\text{where } C_{\mathbb{T}_L}^-(i, j) = (C_{\mathbb{T}_L}(i, j) - C_{\mathbb{T}_L}(j, i))/2. \quad (76)$$

The non-zero commutation relations involving basis elements are thus,

$$[v_1, v_2] = v_3, \quad [v_2, v_3] = v_1, \quad [v_3, v_1] = v_0. \quad (77)$$

Clearly $\text{gen}\{v_0, v_1, v_3\}$ is an ideal of \mathbb{T}_L^- . It is easy to verify that \mathbb{T}_L^- satisfies the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \quad \forall x, y, z \in \mathbb{T}_L^-,$$

and thus, \mathbb{T}_L^- is a **Lie algebra**, and thus \mathbb{T}_L is Lie admissible. It is truly remarkable that this turns out to be a Lie algebra, and that the novel division algebra \mathbb{T}_L can be used to represent it. Furthermore, **the ideal $\text{gen}\{v_0, v_1, v_3\}$ as a subalgebra of \mathbb{T}_L^- is isomorphic to the Heisenberg Algebra. Hence, the algebra \mathbb{T}_L provides a one-dimensional representation of the Heisenberg algebra.** The Lie algebra \mathbb{T}_L^- is thus an extension of the Heisenberg algebra, and it is **solvable** since its commutator (or derived algebra) series becomes $\{0\}$ in finitely many steps:

$$\begin{aligned} \mathcal{D}^0 \mathbb{T}_L^- &= \mathbb{T}_L^-, \quad \mathcal{D}^1 \mathbb{T}_L^- = [\mathcal{D}^0 \mathbb{T}_L^-, \mathcal{D}^0 \mathbb{T}_L^-] = \text{gen}\{v_0, v_1, v_3\}, \\ \mathcal{D}^2 \mathbb{T}_L^- &= [\mathcal{D}^1 \mathbb{T}_L^-, \mathcal{D}^1 \mathbb{T}_L^-] = \text{gen}\{v_0\}, \quad \mathcal{D}^3 \mathbb{T}_L^- = [\mathcal{D}^2 \mathbb{T}_L^-, \mathcal{D}^2 \mathbb{T}_L^-] = \{0\}. \end{aligned}$$

Nevertheless, its lower central series

$$\begin{aligned}\mathbb{T}_{L0}^- &= \mathbb{T}_L^-, & \mathbb{T}_{L1}^- &= [\mathbb{T}_L^-, \mathbb{T}_{L0}^-] = \text{gen}\{v_0, v_1, v_3\}, \\ \mathbb{T}_{Li+1}^- &= [\mathbb{T}_L^-, \mathbb{T}_{Li}^-] = \text{gen}\{v_0, v_1, v_3\} \quad \forall i \geq 1,\end{aligned}$$

never becomes $\{0\}$. So, \mathbb{T}_L^- **is not nilpotent** since it stabilizes in the Heisenberg subalgebra. According to the classification of solvable algebras of dimension four given by W.A. de Graaf, see [30], the algebra \mathbb{T}_L^- corresponds to the 4-dimensional solvable algebra over the reals M_a^{14} , with $a = -1$. This, by the way, proves that this algebra is compatible with a \mathbb{Z}_4 -grading. Although for each $x \in \mathbb{T}_L^-$, the application $y \mapsto [x, y]$ defines a derivation in \mathbb{T}_L^- , it does not define a derivation in \mathbb{T}_L since in general $[x, (y \cdot z)] = [x, y] \cdot z + y \cdot [x, z]$ does not hold. There are interesting inquiries on the classification of real division algebras based on a classification result on the derivation algebras of the real division algebras in [31]. It seems plausible to expect that there might be some information on the division algebra \mathcal{A} encoded in the algebra $(\mathcal{A}^-; +, [\cdot, \cdot])$. We will explore some deformations of the tessaranion algebra below, and we will address some their commutator algebras. A classification of the later might bring some new light into this endeavor.

The tessaranion set with its summation and the **symmetric product** “ \bullet ” in (74) constitutes the algebra \mathbb{T}_L^+ or $(\mathbb{T}_L^+; +, \bullet)$. As a vector space, the algebra \mathbb{T}_L^+ is again generated by $[v_0, v_1, v_2, v_3]$. The product can be given in terms of the structure constants $C_{\mathbb{T}_L}^+$:

$$v_i \bullet v_j = C_{\mathbb{T}_L}^+(i, j) v_{(i+j) \bmod 4}, \quad (78)$$

$$\text{where } C_{\mathbb{T}_L}^+(i, j) = (C_{\mathbb{T}_L}(i, j) + C_{\mathbb{T}_L}(j, i))/2. \quad (79)$$

The non-zero product of basis elements are thus:

$$v_0 \bullet v_i = v_i, \quad \text{for } i = 0, \dots, 3, \quad (80)$$

$$v_1 \bullet v_1 = v_2, \quad v_3 \bullet v_3 = v_2, \quad v_2 \bullet v_2 = -v_0. \quad (81)$$

Clearly, $\text{gen}\{v_0, v_1, v_2\}$ and $\text{gen}\{v_0, v_3, v_2\}$ are ideals of \mathbb{T}_L^+ . It does not satisfy the Jordan identity, since

$$\begin{aligned}(x \bullet y) \bullet (x \bullet x) - x \bullet (y \bullet (x \bullet x)) = \\ (x_1^2 + x_3^2)[-(y_1 x_1 + y_3 x_3), x_1 y_2, 0, x_3 y_2]_{\mathbb{T}_L},\end{aligned} \quad (82)$$

but the identity holds for x being pure even. Accordingly, \mathbb{T}_L^+ **is not a Jordan algebra**. The algebra \mathbb{T}_L^+ **is flexible**, that is $(x \bullet y) \bullet x = x \bullet (y \bullet x)$ for all $x, y \in \mathbb{T}_L^+$, since its product is symmetric. Although $(x \bullet x) \bullet x = x \bullet (x \bullet x)$ for all $x \in \mathbb{T}_L^+$ (since flexible), we have $((x \bullet x) \bullet x) \bullet x \neq (x \bullet x) \bullet (x \bullet x)$ in general. Hence, \mathbb{T}_L^+ **is not power associative**. \mathbb{T}_L^+ **is not alternative** since $(x \bullet x) \bullet y \neq x \bullet (x \bullet y)$, and $(y \bullet x) \bullet x \neq y \bullet (x \bullet x)$, in general.

We summarize these results in the following:

Proposition 4. *The commutator algebra $(\mathbb{T}_L^-; +, [\cdot, \cdot])$ with the product in (73) is a solvable not nilpotent real Lie algebra (77), which provides a one dimensional representation of the Heisenberg algebra, and constitutes an extension of it. The anticommutator algebra $(\mathbb{T}_L^+; +, \bullet)$ with the product in (74) is commutative and thus flexible, but not Jordan, nor alternative, nor power associative algebra. Accordingly, the tesseract algebra \mathbb{T}_L is Lie admissible but not Jordan admissible.*

9 Deformations of the Tesseract Algebra

According to the classical theorem proved independently by Kervaire in [2] and by Bott & Milnor in [3] all finite-dimensional not-necessarily-associative division algebras over the reals have dimensions 1, 2, 4 or 8. It is known, furthermore, that there are infinitely many non-isomorphic of them. In particular, there are deformations of the quaternion algebra depending on a continuous parameter that are not-necessarily-associative division algebras (see [29]). We want to exhibit deformations of the Tesseract algebra \mathbb{T}_L and explore some of their isomorphisms. We use a normalized left-standard basis with the corresponding unital structure constant of \mathbb{Z}_4 in \mathbb{R}^* given in Table IV, but in this case the parameters $\alpha, \beta, \delta, \epsilon, \phi, \omega \in \mathbb{R}^*$, that is, they do not need to take values just in $\{1, -1\}$. We obtain that the determinant of M^L in (21) has the form:

$$\begin{aligned} \det M^L = & y_0^4 + (-\epsilon\beta)y_1^4 + (-\phi)y_2^4 + (-\alpha\delta\omega)y_3^4 + \\ & + (1-\phi)y_2^2 y_0^2 + (\alpha\beta\omega + \epsilon\delta)y_1^2 y_3^2 + \\ & + [(\epsilon(1+\beta) + \phi\beta - 1)y_1^2 + (\alpha(\delta + \phi) - \omega(1-\delta))y_3^2]y_0 y_2 + \\ & + [(\omega - \epsilon\delta - \phi(\alpha\beta - 1))y_2^2 + (\beta\omega)y_3^2]y_0 y_2 \end{aligned} \quad (83)$$

We have already solved the problem of finding not-necessarily-associative division algebras when the parameters $\alpha, \beta, \delta, \epsilon, \phi, \omega$ take values in $\{1, -1\}$. Although we consider now a more general situation, we will proceed in a similar manner. We consider first the case when two components are zero and analyze the determinant. For $y_2 = y_3 = 0$ we obtain the necessary condition for absence of zero divisors $-\epsilon\beta > 0$. For $y_1 = y_2 = 0$ we obtain the necessary condition $-\alpha\delta\omega > 0$. For $y_1 = y_3 = 0$ we obtain the determinant:

$$\det M^L|_{y_1=y_3=0} = y_0^4 + (1-\phi)y_2^2 y_0^2 - \phi y_2^4.$$

We assume $y_2 \neq 0$ and solve $\det M^L/y_2^4 = 0$ for $(y_0/y_2)^2$ to obtain:

$$(y_0/y_2)^2 = \frac{-(1-\phi) \pm |1+\phi|}{2}.$$

We conclude that there are no real solutions when the right-hand side is always negative, which can only happen when $-\phi > 0$. For $y_0 = y_2 = 0$ we obtain

$$\det M^L|_{y_0=y_2=0} = (-\epsilon\beta)y_0^4 + (\alpha\beta\omega + \epsilon\delta)y_1^2 y_3^2 + (-\alpha\delta\omega)y_3^4.$$

We assume $y_3 \neq 0$ and solve $\det M^L/y_3^4 = 0$ for $(y_1/y_3)^2$ to obtain:

$$(y_1/y_3)^2 = \frac{-(\alpha\beta\omega + \epsilon\delta) \pm |\alpha\beta\omega - \epsilon\delta|}{2(-\epsilon\beta)}.$$

In order to have negative right-hand side (and thus no real solution for y_1/y_3) we need to require –beyond the constraints already obtained– that $\alpha\beta\omega > 0$, and $\epsilon\delta > 0$. We summarize the constraints obtained

$$-\epsilon\beta > 0, \quad -\alpha\delta\omega > 0, \quad -\phi > 0, \quad \alpha\beta\omega > 0, \quad \epsilon\delta > 0. \quad (84)$$

Three of these results can be obtained immediately with the following argument. The determinant in (83) when all but one component are zero has one of the forms y_0^4 , $(-\epsilon\beta)y_1^4$, $(-\phi)y_2^4$, $(-\alpha\delta\omega)y_3^4$. Now, since the determinant in (83) is an overall continuous polynomial function, if the determinants for a single non-zero component had different signs there would be some point where the determinant gets the value zero (passing, say, from positive to negative values). Hence, all the determinants have to have necessarily the same sign, leading to three of the obtained constraints. From the necessary constraints (84) it follows also:

$$-\delta\beta > 0, \quad -\alpha\epsilon\omega > 0.$$

Observe, that using the obtained constraints we could reduce the search of nonsymmetric division algebras when the structure constant C of \mathbb{Z}_4 takes values in $\{1, -1\}$ to only four cases. In order to explore the multiparametric region of deformed \mathbb{T}_L -algebras, we will fix all the parameters but some few in the values they adopt for the \mathbb{T}_L algebra, and look for conditions to obtain not-necessarily-associative division algebras.

(a) The algebra $\mathbb{T}_L^1(k)$ with $k \neq 1$, $k > 0$:

We adopt a structure constant C given in table IV, with

$$\alpha = -k, \quad \beta = -1, \quad \delta = 1, \quad \epsilon = 1, \quad \phi = -1, \quad \omega = k, \quad (85)$$

and $k \neq 1$, $k > 0$. We solve $\det M^L = 0$ for y_2 , and $\det M^R = 0$ for x_2 to obtain, respectively,

$$\begin{aligned} y_2 &= \pm \sqrt{-y_0^2 + (1-k)y_1 y_3 \pm (i y_1^2 + i k y_3^2)}, \\ x_2 &= \pm \sqrt{-x_0^2 \pm (x_1^2 + x_3^2) \sqrt{-k}}, \end{aligned}$$

where, in each case the two instances of sign choices are independent (leading to four sign choices and up to four solutions). For $k > 0$ we see that there are no non-trivial real solutions for $\det M^L = \det M^R = 0$, and thus no zero divisors. The algebras $\mathbb{T}_L^1(k)$, with $k \neq 1$, $k > 0$, are not isomorphic to \mathbb{T}_L since we already found that the automorphisms of \mathbb{T}_L for the adopted basis choice leads always to the single structure constant $C_{\mathbb{T}_L}$.

We investigate now the algebra automorphism of $\mathbb{T}_L^1(k)$, $k > 0$. We adopt a new basis of $\mathbb{T}_L^1(k)$ as a vector space: $\{1, v'_1, v_1'^2, v'_1 \cdot (v_1'^2)\}$, for $v'_1 = [a_0, a_1, a_2, a_3]$. We require that $(v_1'^2) \cdot (v_1'^2) = [-1, 0, 0, 0]$, and that $v'_1 \cdot (v_1' \cdot v_1'^2) = [\rho, 0, 0, 0]$, for some $\rho \in \mathbb{R}$. We obtain, for $k \neq 1$ that either $v'_1 = [0, \pm 1, 0, 0]$ and $\rho = k$ (which leads to the same structure constant already adopted for $\mathbb{T}_L^1(k)$, $k > 0$: Table IV with parameter choices (85)), or $v'_1 = [0, 0, 0, \pm 1/\text{sqrt}(k)]$. With this later choice, and the left-standard basis we obtain a structure constant given by table IV, with the choices:

$$\alpha = -1/k, \beta = -1, \delta = 1, \epsilon = 1, \phi = -1, \omega = 1/k.$$

But this is the same choice as in (85) when changing $k > 0$ by $1/k > 0$. Hence, for different values of $0 < k < 1$ we obtain non-isomorphic division algebras and thus we have non-countable infinitely many not-necessarily-associative division algebras. This gives an alternative proof of a known fact:

Proposition 5. *There are non-countable infinitely many non-isomorphic not-necessarily-associative division algebras.*

Observe that the algebras $\mathbb{T}_L^1(k)$ for $0 < k \neq 1$ does not seem to be included in the Cayley–Dickson doublings considered in [8]. We do not determine if they are included in the generalized Cayley–Dickson doublings considered in [7], and [4].

The set $\mathbb{T}_L^1(k)$ with its summation and the product “[\cdot, \cdot]” in (73) constitutes the algebra $(\mathbb{T}_L^1(k)^-, +, [\cdot, \cdot])$. The non-trivial commutation relations between its generators are:

$$[v_1, v_2] = v_3, \quad [v_2, v_3] = v_1, \quad [v_3, v_1] = \frac{1+k}{2} v_0.$$

This is isomorphic to the Lie algebra $(\mathbb{T}_L^-, +, [\cdot, \cdot])$ using a simple change of variables, say $v'_1 = u v_1, v'_2 = u v_2$, for an u satisfying $u^{-2} = \frac{1+k}{2}$.

(b) **Algebra $\mathbb{T}_L^2(k)$ for $k \neq 1$, $k > 0$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -1, \beta = -k, \delta = k, \epsilon = 1, \phi = -1, \omega = 1, \quad (86)$$

and $k \neq 1$, $k > 0$. We solve $\det M^L = 0$ for y_2 to obtain

$$y_2 = \pm \sqrt{-y_0^2 \pm (y_1^2 + y_3^2) \sqrt{-k}}, \quad (87)$$

where again, the two instances of sign choices are independent. For $k > 0$ we see that there are no non-trivial real solutions for $\det M^L = 0$. There are thus no zero divisors since absence of left zero divisors implies absence of right zero divisors.

(c) **Algebra $\mathbb{T}_L^3(k)$ for $k \neq 1, k > 0$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -1, \beta = -1, \delta = 1, \epsilon = k, \phi = -1, \omega = k, \quad (88)$$

and $k \neq 1, k > 0$. We solve $\det M^L = 0$ for y_2 to obtain the same solution as (87). This leads again to the absence of zero divisors.

(d) **Algebra $\mathbb{T}_L^4(k)$ for $k \neq 1, k > (2/3)\sqrt{3} - 1$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -k, \beta = -1, \delta = 1, \epsilon = k, \phi = -1, \omega = k, \quad (89)$$

and $k \neq 1, k > (2/3)\sqrt{3} - 1$. We solve $\det M^L = 0$ for y_2 to obtain

$$y_2 = \pm \frac{1}{2} \sqrt{-4y_0^2 + 2(1-k)y_1y_3 \pm 2\sqrt{P(y_1, y_3)}}, \quad (90)$$

$$\text{where } P(y_1, y_3) = -(3k^2 + 6k - 1)y_1^2y_3^2 - 4k^2y_3^4 - 4ky_1^4.$$

We note that for $k > (2/3)\sqrt{3} - 1$ we have $(3k^2 + 6k - 1) > 0$. And thus, unless $y_1 = y_3 = 0$, the radical inside the radical in (90) is imaginary. Now, if $y_1 = y_3 = 0$, then y_2 is also imaginary, unless $y_0 = 0$. Accordingly, for $k > (2/3)\sqrt{3} - 1$ there are no zero divisors.

(e) **Algebra $\mathbb{T}_L^5(k)$ for $k \neq 1, 0 < k \leq 3 + 2\sqrt{3}$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -1, \beta = -1, \delta = k, \epsilon = 1, \phi = -1, \omega = 1, \quad (91)$$

and $k \neq 1, 0 < k \leq 3 + 2\sqrt{3}$. We solve $\det M^L = 0$ for y_2 to obtain

$$y_2 = \pm \frac{1}{2} \sqrt{-4y_0^2 + 2(1-k)y_1y_3 \pm 2\sqrt{Q(y_1, y_3)}}, \quad (92)$$

$$\text{where } Q(y_1, y_3) = -(3 + 6k - k^2)y_1^2y_3^2 - 4ky_3^4 - 4ky_1^4.$$

We note that for $0 < k \leq 3 + 2\sqrt{3}$ we have $(3 + 6k - k^2) > 0$. And thus, unless $y_1 = y_3 = 0$, the radical inside the radical in (92) is imaginary. Now, if $y_1 = y_3 = 0$, then y_2 is also imaginary, unless $y_0 = 0$. Accordingly, for $0 < k \leq 3 + 2\sqrt{3}$ there are no zero divisors.

Remark: For $-(3 + 6k - k^2) < 4\sqrt{k}$ and $y_1y_3 \neq 0$, we have $Q(y_1, y_3) < 4\sqrt{k}y_1^2y_3^2 - 4k(y_3^2)^2 - 4(y_1^2)^2 = -(2\sqrt{k}y_3^2 - 2y_1^2)^2 \leq 0$. Hence, for $4\sqrt{k} - k^2 + 6k + 3 > 0$ we will have non-real radical inside the radical in (92) when $y_1y_3 \neq 0$. Now, If $y_1y_3 = 0$ we obtain also non-real y_2 in (92) unless $y_0 = 0$. Hence, for $0 < k < \rho$, where ρ is a positive root of $4\sqrt{\rho} - \rho^2 + 6\rho + 3 = 0$ (which has approximately the value $\rho \sim 7.81$). This leads to a larger interval than $0 < k \leq 3 + 2\sqrt{3}$. Similar refinements might be possible for some of the other cases.

(f) **Algebra $\mathbb{T}_L^6(k)$ for $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -1, \beta = -1, \delta = 1, \epsilon = 1, \phi = -1, \omega = k, \quad (93)$$

and $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$. We solve $\det M^L = 0$ for y_2 to obtain the same solution as in (92). Hence, there are no zero divisors.

(g) **Algebra $\mathbb{T}_L^7(k)$ for $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -1, \beta = -k, \delta = 1, \epsilon = 1, \phi = -1, \omega = 1, \quad (94)$$

and $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$. We solve $\det M^L = 0$ for y_2 to obtain the same solution as in (92) but exchanging y_1 and y_3 . Hence, there are no zero divisors.

(h) **Algebra $\mathbb{T}_L^8(k)$ for $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$:**

We adopt a structure constant C given in table IV, with

$$\alpha = -k, \beta = -1, \delta = 1, \epsilon = 1, \phi = -1, \omega = 1, \quad (95)$$

and $k \neq 1$, $0 < k \leq 3 + 2\sqrt{3}$. We solve $\det M^L = 0$ for y_2 to obtain (again) the same solution as in (92) but exchanging y_1 and y_3 . Hence, there are no zero divisors.

The classification of twisted group division algebras whose structure constant takes values in a larger multiplicative subgroup of \mathbb{R}^* is far from complete. Clearly, even if we start with a structure constant in $\{1, -1\}$, a change of basis might lead to a structure constant not in $\{1, -1\}$. We classified in previous sections the twisted group division \mathbb{R} -algebras, of grading group G of order $|G| \leq 4$ which allow for a basis in which the structure constant is in $\{1, -1\}$. Nevertheless, from the considerations above, there seems to exist a multi-parametric region satisfying the obtained overall constraints (84) that leads to NNA twisted group division \mathbb{R} -algebras. The parameter choices leading to non isomorphic NNA division algebras will define a point in a moduli space, whose coordinates give the parameters for non isomorphic NNA twisted group division \mathbb{R} -algebras. We call such algebras “deformations” of the one already found with structure constant in $\{1, -1\}$, which is clearly included in the region. We just explored here some one-parametric cases, and determined that there are non countable many non isomorphic twisted group division algebras in the region. Nevertheless, we could conjecture that each set of “deformations” might lead to few (perhaps a single) non-isomorphic commutator Lie algebras. We will study in [41] a more general classification of twisted group division \mathbb{R} -algebras with generic unital structure constants in \mathbb{R}^* . An open question will be addressed there is whether these deformation algebras could be represented in terms of matrices with entries in the novel division algebras which allow for a basis with

unital structure constants in $\{1, -1\}$. Observe for now, that some deformations of the quaternion algebra lead to nonassociative division algebras, which clearly cannot be represented through standard quaternionic (square) matrix arrays since such matrix algebras are associative.

10 Conclusions and further applications

We have classified the twisted group algebras $(\mathcal{A}; G, K, A, C)$ of the form $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ which are not-necessarily-associative division algebras over \mathbb{R} and have grading group of order $|G| \leq 4$. This provides an extremely effective frame for the determination of the Cayley-Dickson division algebra series \mathbb{C} , \mathbb{H} , (and octonion) algebras, but it also provides new algebras non-isomorphic to the known series. In particular, a novel not associative and not commutative division \mathbb{R} -algebra of dimension four has been found. The \mathbb{R} -algebra \mathbb{C} could be characterized for solving the equation $x^2 + 1 = 0$ (clearly beyond the possibilities of \mathbb{R}), and with it providing an algebraically closed field. The quaternions can solve the equation in two variables $(xy - yx)^2 + 4 = 0$ (clearly beyond the possibilities of the complex field). Now, the novel not associative and not commutative division \mathbb{R} -algebra of tessaranions \mathbb{T}_L can solve the equation $(x \cdot x^2 - x^2 \cdot x)x - 2 = 0$ which can not be solved in any alternative \mathbb{R} -algebra. Observe that the identities leading to alternativity can be used to prove the existence of “parenthesized” polynomial equations (emphasizing that the position of parenthesis does matter) of degree higher than zero that can not be solved by any alternative \mathbb{R} -algebra. We presented here some of the universal polynomial identities satisfied by the tessaranions, under the name of “tesseraninity”. Observe that the tessaranions can not solve the “parenthesized” equation $x \cdot (x \cdot x^2) + (x^2 \cdot x) \cdot x - 2x^2 \cdot x^2 + 1 = 0$ since it obeys the quartic tesseraninity identity (44).

In [19] we will classify the twisted group division algebras $(\mathcal{A}; G, \mathbb{R}, \{1, -1\}, C)$ graded over the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_4$. The paper [20] classifies those graded by the dihedral group D_4 and those graded by the quaternion group. The last contribution of this series is [21], which will address those graded by \mathbb{Z}_8 .

There is also a paper series [41] in preparation devoted to applications of the novel algebras. The question about universal polynomial identities satisfied by the novel not associative division \mathbb{R} -algebras will be presented there, some of them under the name of “tesseraninity” identities (which are an analogous to alternativity identities). This paper series addresses also the meaning of algebraic closedness in not associative \mathbb{R} -algebras. It addresses also the representation theory of some loops and some non-associative algebras, maps between spheres and Hopf fibrations, generalized Cayley-Dickson process and its application over fields of non-zero characteristic, projective spaces and its applications to error correcting codes and cryptography. We study also matrix algebras in particular self-adjoint matrix algebras over the novel not associative division algebras. We study the analysis on functions over the new not associative division algebras, and some applications to differential equations. It studies also discrete versions

of the obtained algebras (analogous to Lipschitz and Hurwitz integer quaternions, and octavian integer octonions, see [34]) and the exploration of novel power reciprocities and power residues.

In a different venue, a relation has been established between the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and octonions, their corresponding triality maps, with spinors and super Yang–Mills theories (and super-gravity theories) in 3, 4, 6, or 10 Minkowski space-time dimensions, see [35] and references therein. The author has found a novel $\mathbb{Z}_4 \times \mathbb{Z}_4$ -graded extension of the Poincaré algebra called the clover extension, see [36]–[40], which is quite analogous to supersymmetry. This motivated also the present exploration. The task was to find the division algebras behind novel space-time symmetries. The obtained NNA division algebras have passed the reciprocal task of finding the space-time symmetries related to them, which will be addressed in [41].

11 Appendix A

We will determine a family of norms, which are defined iteratively, through the following result:

Theorem 2. *We consider n -tuples of the form:*

$$\begin{aligned} a^{(1)} &= (a_1, a_2, \dots, a_n) \quad , \quad a^{(2)} = (a_{n+1}, a_{n+2}, \dots, a_{2n}), \dots, \\ a^{(m)} &= (a_{n(m-1)+1}, a_{n(m-1)+2}, \dots, a_{nm}). \end{aligned}$$

The following functions M_j for $j = 1, 2, 3, \dots$, are norms, where

$$M_1(a^{(1)}) = (a_1^2 + a_2^2 + \dots + a_n^2)^{\frac{1}{2}}, \quad (96)$$

$$\begin{aligned} M_2((a^{(1)}, a^{(2)})) &= (M_1(a^{(1)})^4 + M_1(a^{(2)})^4)^{\frac{1}{4}} \\ &= ((a_1^2 + \dots + a_n^2)^2 + (a_{n+1}^2 + \dots + a_{2n}^2)^2)^{\frac{1}{4}}, \end{aligned} \quad (97)$$

$$M_3(((a^{(1)}, a^{(2)}), (a^{(3)}, a^{(4)}))) = (M_2((a^{(1)}, a^{(2)}))^8 + M_2((a^{(3)}, a^{(4)}))^8)^{\frac{1}{8}}, \quad (98)$$

$$\vdots \quad \vdots$$

$$M_j((u, r)) = (M_{j-1}(u)^{2^j} + M_{j-1}(r)^{2^j})^{\frac{1}{2^j}}. \quad (99)$$

Proof. It is clear that M_1, M_2, M_3, \dots fulfil positive homogeneity and positive definiteness:

$$M_j(\alpha x) = |\alpha| M_j(x), \forall \alpha \in \mathbb{R}, x \in \mathbb{R}^{n \cdot 2^{j-1}} \text{ (positive homogeneity)}, \quad (100)$$

$$M_j(x) = 0 \implies x = 0 \quad \forall x \in \mathbb{R}^{n \cdot 2^{j-1}} \text{ (positive definiteness)}. \quad (101)$$

To prove

$$M_j(x + y) \leq M_j(x) + M_j(y), \quad \forall x, y \in \mathbb{R}^{n \cdot 2^{j-1}} \text{ (triangle inequality)}, \quad (102)$$

we will proceed by induction. We quote first a particular version of Holder's inequality:

$$\begin{aligned} |\rho_1||\xi_1| + |\rho_2||\xi_2| &\leq (|\rho_1|^p + |\rho_2|^p)^{\frac{1}{p}} (|\xi_1|^q + |\xi_2|^q)^{\frac{1}{q}}, \quad (103) \\ \text{for } \rho_1, \rho_2, \xi_1, \xi_2 \in \mathbb{R}, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \text{and } p, q \in \mathbb{R}^+. \end{aligned}$$

The function M_1 is clearly an Euclidean norm. From the triangle inequality for M_1 ,

$$\begin{aligned} M_1(a^{(1)} + b^{(1)}) &= ((a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2)^{\frac{1}{2}} = \\ &\leq (a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}} + (b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}}. \end{aligned}$$

Equivalently,

$$M_1(a^{(1)} + b^{(1)})^4 \leq (M_1(a^{(1)}) + M_1(b^{(1)}))^4. \quad (104)$$

Hence, from (97),

$$\begin{aligned} &M_2((a^{(1)} + b^{(1)}, a^{(2)} + b^{(2)}))^4 \\ &= M_1(a^{(1)} + b^{(1)})^4 + M_1(a^{(2)} + b^{(2)})^4 \\ &\leq (M_1(a^{(1)}) + M_1(b^{(1)}))^4 + (M_1(a^{(2)}) + M_1(b^{(2)}))^4 \\ &= M_1(a^{(1)})^4 + M_1(a^{(2)})^4 + M_1(a^{(2)})^4 + M_1(b^{(2)})^4 \\ &\quad + 4[M_1(a^{(1)})^3 M_1(b^{(1)}) + M_1(a^{(2)})^3 M_1(b^{(2)})] \\ &\quad + 6[M_1(a^{(1)})^2 M_1(b^{(1)})^2 + M_1(a^{(2)})^2 M_1(b^{(2)})^2] \\ &\quad + 4[M_1(a^{(1)}) M_1(b^{(1)})^3 + M_1(a^{(2)}) M_1(b^{(2)})^3]. \end{aligned} \quad (105)$$

If we take $|\rho_i| = M_1(a^{(i)})^3$, $|\xi_i| = M_1(b^{(i)})$, for $i = 1, 2$ and use Holder's inequality (103) for $p = \frac{4}{3}$ and $q = \frac{4}{1}$ in the first factor in square brackets in (105), we obtain

$$\begin{aligned} &[M_1(a^{(1)})^3 M_1(b^{(1)}) + M_1(a^{(2)})^3 M_1(b^{(2)})] \\ &\leq (M_1(a^{(1)})^4 + M_1(a^{(2)})^4)^{\frac{3}{4}} (M_1(b^{(1)})^4 + M_1(b^{(2)})^4)^{\frac{1}{4}} \\ &= M_2((a^{(1)}, a^{(2)}))^3 M_2((b^{(1)}, b^{(2)})). \end{aligned} \quad (106)$$

Similar inequalities are obtained for the second and third factor in square brackets in (105), and we can write finally:

$$\begin{aligned} &M_2((a^{(1)} + b^{(1)}, a^{(2)} + b^{(2)}))^4 \\ &\leq M_2((a^{(1)}, a^{(2)}))^4 + M_2((b^{(1)}, b^{(2)}))^4 \\ &\quad + 4[M_2((a^{(1)}, a^{(2)}))^3 M_2((b^{(1)}, b^{(2)}))] \\ &\quad + 6[(M_2((a^{(1)}, a^{(2)}))^2 M_2((b^{(1)}, b^{(2)}))^2] \\ &\quad + 4[M_2((a^{(1)}, a^{(2)})) M_2((b^{(1)}, b^{(2)}))^3] = \\ &= [M_2((a^{(1)}, a^{(2)})) + M_2((b^{(1)}, b^{(2)}))]^4, \end{aligned} \quad (107)$$

from which the validity of the triangle inequality (102) for $j = 2$ follows. We assume now (induction hypothesis) that the triangle inequality (102) holds for M_j . In particular,

$$M_j(u + v)^{2^{j+1}} \leq (M_j(u) + M_j(v))^{2^{j+1}}. \quad (108)$$

Using the result above, the function M_{j+1} fulfills

$$\begin{aligned} M_{j+1}((u + v, w + x))^{2^{j+1}} &= M_j(u + v)^{2^{j+1}} + M_j(w + x)^{2^{j+1}} \\ &\leq (M_j(u) + M_j(v))^{2^{j+1}} + (M_j(w) + M_j(x))^{2^{j+1}} \\ &= M_j(u)^{2^{j+1}} + M_j(v)^{2^{j+1}} + M_j(w)^{2^{j+1}} + M_j(x)^{2^{j+1}} \\ &\quad + 2^{j+1}[M_j(u)^{2^{j+1}-1} M_j(v) + M_j(w)^{2^{j+1}-1} M_j(x)] + \dots \\ &\quad + 2^{j+1}[M_j(u) M_j(v)^{2^{j+1}-1} + M_j(w) M_j(x)^{2^{j+1}-1}] \end{aligned} \quad (109)$$

$$\begin{aligned} &\leq M_{j+1}((u, w))^{2^{j+1}} + M_{j+1}((v, x))^{2^{j+1}} \\ &\quad + 2^{j+1}[M_{j+1}((u, w))^{2^{j+1}-1} M_{j+1}((v, x))] + \dots \\ &\quad + 2^{j+1}[M_{j+1}((u, w)) M_{j+1}((v, x))^{2^{j+1}-1}] \end{aligned} \quad (110)$$

$$= [M_{j+1}((u, w)) + M_{j+1}((v, x))]^{2^{j+1}}, \quad (111)$$

where we used the definition of M_{j+1} in (99), and the Holder's inequality for each square bracketed term in order to move from (109) to (110). \square

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